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RESEARCH ARTICLE

The odd Chen generator of distributions: properties and estimation methods with applications in medicine and engineering

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Abstract: This paper introduces a new univariate flexible generator of distributions called the odd Chen-G family, and some of its statistical properties are derived. Two special models of the proposed generator are provided. The model parameters are estimated using six estimation methods, namely, maximum likelihood estimators, least squares estimators, weighted least squares estimators, maximum product of spacings estimators, Cramér-von Mises estimators and percentile based estimators. Further, simulations are performed to compare their performances for both small and large samples. Finally, two real datasets are used to illustrate the flexibility of the special models of the proposed family.

Keywords: Chen distribution, estimation methods, hazard rate function, maximum likelihood estimators, simulation, weighted least squares estimators.

INTRODUCTION

Numerous classical distributions have been extensively used over the past decades for modelling data in several areas such as medical sciences, life testing problems, biological studies, demography, engineering, actuarial, environmental, and economics. However, in many applied areas such as insurance, survival and reliability theory, there is a clear need for extended forms of these distributions, because, in many practical situations, classical distributions do not provide adequate fits to real data. Therefore, there has been an increased interest in developing more flexible distributions; for example, El-Gohary *et al*. (2015a, 2015b), El-Bassiouny *et al.* (2016, 2017), El-Morshedy *et al*. (2017), Eliwa *et al*. (2018) and Jehhan *et al*. (2018) among others.

Recently, several families employing one or more parameters to generate new distributions have been proposed in the literature; for example, Alzaatreh *et al.* (2013), Amini *et al*. (2018), Haghbin *et al*. (2016), El-Morshedy and Eliwa (2019) and Cordeiro *et al*. (2019), Alizadeh *et al*. (2020), among others.

Chen (2000) proposed a new two-parameter distribution with bathtub shaped or increasing failure rate function called the Chen (Ch) distribution with positive parameters α and β. The cumulative distribution function (CDF) and probability density function (PDF) of the Ch distribution are

$$\Pi(t;\alpha,\beta) = 1 - \exp(-\alpha\left[\exp\left(t^{\beta}\right) - 1\right]),\ t > 0 \qquad ...(01)$$

and

$$\pi(t;\alpha,\beta) = \alpha\beta t^{\beta-1}\exp\left(t^{\beta}\right)\exp(-\alpha\left[\exp\left(t^{\beta}\right) - 1\right]),\ t > 0. \qquad ...(02)$$

Several extensions of the Ch distribution are discussed in the statistical literature; see for example Chaubey and Zhang (2015), Dey *et al*. (2017) and Shuaib *et al*. (2018),

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among others. Due to the importance of the Ch distribution, we introduce a new class called the odd Chen-G (OCh-G) family of distributions.

METHODOLOGY

Assume T is the lifetime of a system following the Ch CDF in equation (1). If the random variable X represents the odds ratio, the risk that this system will not be working at time x is given by $G(x;\eta)/[1-G(x;\eta)]$, where $G(x;\eta)$ represents the CDF of a baseline model with parameter vector $\eta\ (1\times k)$.

$$F(x)=P(X\leq x)=\Pi\left(\frac{G(x;\eta)}{1-G(x;\eta)}\right),$$

where the odds ratio $G(x;\eta)/[1-G(x;\eta)]$ satisfies the following conditions:

1. $G(x;\eta)/[1-G(x;\eta)]\in[a^*,b^*]$ for $0<a^*<b^*<\infty$.
2. $G(x;\eta)/[1-G(x;\eta)]$ is differentiable and monotonically non-decreasing.
3. $G(x;\eta)/[1-G(x;\eta)]\to a^*$ as $x\to 0$ but $G(x;\eta)/[1-G(x;\eta)]\to b^*$ as $x\to\infty$.

Let $g(x;\eta)$, $G(x;\eta)$, and $\overline{G}(x;\eta)$, respectively denote the PDF, CDF and RF of a baseline model with parameter vector $\eta\ (1\times k)$. Then, the CDF of the OCh-G family is given by

$$F(x;\alpha,\beta,\eta)=\int_0^{\frac{G(x;\eta)}{1-G(x;\eta)}}\pi(t)dt$$

$$=1-\exp\left(-\alpha\left\{\exp\left[\frac{G(x;\eta)}{1-G(x;\eta)}\right]^{\beta}-1\right\}\right). \quad ...(03)$$

The PDF of the OCh-G family is given as follows

$$f(x;\alpha,\beta,\eta)=\frac{\alpha\beta g(x;\eta)G(x;\eta)^{\beta-1}}{[1-G(x;\eta)]^{\beta+1}}\exp\left[\frac{G(x;\eta)}{1-G(x;\eta)}\right]^{\beta}$$

$$\exp\left(-\alpha\left\{\exp\left[\frac{G(x;\eta)}{1-G(x;\eta)}\right]^{\beta}-1\right\}\right). \quad ...(04)$$

Using the power series for the exponential function and the generalised binomial expansion, equations (3) and (4) can be represented as an innite mixture of exponential-G (Exp-G) density functions as follows

$$F(x;\alpha,\beta,\eta)=\sum_{i=1}^{\infty}\sum_{j=0}^{i}\sum_{k,m=0}^{\infty}\Upsilon_{i,j,k,m}(\alpha,\beta)H_{\beta k+m}(x), \quad ...(05)$$

and

$$f(x;\alpha,\beta,\eta)=\sum_{i=1}^{\infty}\sum_{j=0}^{i}\sum_{k,m=0}^{\infty}\Upsilon_{i,j,k,m}(\alpha,\beta)h_{\beta k+m}(x), \quad ...(06)$$

respectively, where $H_{\beta k+m}(x)=G(x;\eta)^{\beta k+m}$ is the $Exp-G$ family with power parameter $(\beta k+m)$, $h_{\beta k+m}(x)=(\beta k+m)g(x;\eta)G(x;\eta)^{\beta k+m-1}$, and

$$\Upsilon_{i,j,k,m}(\alpha,\beta)=\frac{(-1)^{i+j}\,\alpha^i\,(i-j)^k\,\Gamma(\beta k+m)}{i!\ k!\ m!\Gamma(\beta k)}C_j^i,$$

where $(\beta k)\neq 0,-1,-2,-3,\ldots$ and $(\beta k+m)\neq 0,-1,-2,-3,\ldots$. The reliability function (RF) of the OCh-G class is

$$R(x;\alpha,\beta,\eta)=\exp\left(-\alpha\left\{\exp\left[\frac{G(x;\eta)}{1-G(x;\eta)}\right]^{\beta}-1\right\}\right);\quad x>0. \quad ...(07)$$

The hazard rate function (HRF) is

$$h(x;\alpha,\beta,\eta)=\frac{\alpha\beta g(x;\eta)G(x;\eta)^{\beta-1}}{[1-G(x;\eta)]^{\beta+1}}\exp\left(\frac{G(x;\eta)}{1-G(x;\eta)}\right);\quad x>0. \quad ...(08)$$

There are following motivations for introducing the OCh-G family:

i. To construct heavy-tailed distributions for modelling real data.
ii. To define special models with all types of the HRF.
iii. To make the kurtosis more flexible compared to the baseline model.
iv. To provide consistently better fits than other generated models under the same baseline distribution.

v. To generate distributions with symmetric, left-skewed and right-skewed shaped.
vi. To study which is the best method to estimate the model parameters.

Furthermore, we are also motivated to illustrate how different estimators of the special sub models of the OCh-G class perform for various sample sizes and various parameter combinations and to develop a guideline for determining the best method of estimation, which is important for the applied statisticians.

Statistical properties

Asymptotics

Let $a = \inf\{x|G(x;\eta) > 0\}$, then the asymptotics of the CDF, PDF and HRF when $x \to a$ are, respectively, given by

$$F(x;\alpha,\beta,\eta) \sim \alpha\, G(x;\eta)^{\beta}$$

$$f(x;\alpha,\beta,\eta) \sim \alpha\beta\, g(x;\eta)\; G(x;\eta)^{\beta-1}$$

and

$$h(x;\alpha,\beta,\eta) \sim \alpha\beta\, g(x;\eta)\; G(x;\eta)^{\beta-1}.$$

Further, the asymptotics of the CDF, PDF and HRF when $x \to \infty$ are, respectively, given by

$$1 - F(x;\alpha,\beta,\eta) \sim e^{-\alpha e^{[1-G(x;\eta)]^{-\beta}}},$$

$$f(x;\alpha,\beta,\eta) \sim \alpha\beta\, g(x;\eta)\,[1-G(x;\eta)]^{-\beta-1} e^{[1-G(x;\eta)]^{-\beta}} e^{-\alpha e^{[1-G(x;\eta)]^{-\beta}}}$$

$$h(x;\alpha,\beta,\eta) \sim \alpha\beta\, g(x;\eta)\,[1-G(x;\eta)]^{-\beta-1} e^{[1-G(x;\eta)]^{-\beta}}.$$

Quantile function

Assume $X \sim$ OCh-G family, for any $u^* \in (0,1)$, the u^*th quantile function, say $Q(u^*)$, of X is the solution of $F(Q(u^*)) - u^* = 0$; $Q(u^*) > 0$, then

$$Q(u^*) = G^{-1}\left(\frac{1}{1+\left\{\log\left(1-\frac{1}{\alpha}\log\left[1-u^*\right]\right)\right\}^{-1/\beta}}\right), \quad ...(09)$$

where G^{-1} represents the baseline quantile function. Setting $u^* = 0.5$, we get the median of X.

Moments, skewness, kurtosis and mean deviations

If $X \sim$ OCh-G family, the r^{th} moment of X is given by

$$\begin{aligned}\mu_r^{'} = \mathbf{E}(X^r) &= \int_0^{\infty} x^r f(x;\alpha,\beta,\eta)dx \\ &= \sum_{i=1}^{\infty}\sum_{j=0}^{i}\sum_{k,m=0}^{\infty} \Upsilon_{i,j,k,m}(\alpha,\beta)\int_0^{\infty} x^r h_{\beta k+m}(x)dx \\ &= \sum_{i=1}^{\infty}\sum_{j=0}^{i}\sum_{k,m=0}^{\infty} \Upsilon_{i,j,k,m}(\alpha,\beta)E(Y_{\beta k+m}^r),\end{aligned} \quad ...(10)$$

where $Y_{\beta k+m}^r \sim Exp-G$ with power parameter $(\beta k+m)$. Setting $r = 1$ in equation (10), we obtain the mean of X. Moreover, we can derive an important measure in survival analysis called the mean time to failure (MTTF) where $MTTF = \mu_1^{'}$. This measure can be used in order to design and manufacture a maintainable system.

On the other hand, the skewness and the kurtosis can be calculated, respectively, as follows:

$$\delta_1 = \left(\mu_3^{'} - 3\mu_2^{'}\mu_1^{'} + 2\mu_1^{'3}\right) / \left(\mu_2^{'} - \mu_1^{'2}\right)^{3/2} \quad ...(11)$$

and

$$\delta_2 = \left(\mu_4^{'} - 4\mu_3^{'}\mu_1^{'} + 6\mu_2^{'}\mu_1^{'2} - 3\mu_1^{'4}\right) / \left(\mu_2^{'} - \mu_1^{'2}\right)^2. \quad ...(12)$$

Further, the incomplete moments play an important role for measuring inequality. For example, the first incomplete moment can be used to obtain the formulas of Lorenz and Bonferroni curves. The qth incomplete moment of X can be expressed as follows:

$$\begin{aligned}\Phi_{(q)}(t) &= \int_0^t x^q f(x;\alpha,\beta,\eta)dx \\ &= \sum_{i=1}^{\infty}\sum_{j=0}^{i}\sum_{k,m=0}^{\infty} \Upsilon_{i,j,k,m}(\alpha,\beta)\int_0^t x^q h_{\beta k+m}(x)dx \\ &= \sum_{i=1}^{\infty}\sum_{j=0}^{i}\sum_{k,m=0}^{\infty} \Upsilon_{i,j,k,m}(\alpha,\beta)\Phi_{(q)}^{*}(t),\end{aligned} \quad ...(13)$$

where $\Phi^*_{(q)}(t)=\int_0^t x^q h_{\beta k+m}(x)dx$. On the other hand, the mean deviations about the mean and the median can be represented as

$$\lambda_1 = 2\mu_1^{'}F(\mu_1^{'}) - 2\Phi_{(1)}(\mu_1^{'})$$

and

$\lambda_2 = \mu_1^{'} - 2\Phi_{(1)}(Q(0.5))$, respectively.

Bonferroni and Lorenz curves

Bonferroni (1930) presented the Bonferroni and Lorenz curves. These curves have applications in reliability, medicine, demography, economics and insurance. If $X \sim$OCh-G family, then the Bonferroni curve is given by

$$\begin{aligned} B(p) &= \frac{1}{\mu^* p}\int_0^{G^{-1}(p)} x f(x;\alpha,\beta,\eta)dx \\ &= \frac{1}{\mu^* p}\sum_{i=1}^{\infty}\sum_{j=0}^{i}\sum_{k,m=0}^{\infty}\Upsilon_{i,j,k,m}(\alpha,\beta)\int_0^{G^{-1}(p)} x h_{\beta k+m}(x)dx \\ &= \sum_{i=1}^{\infty}\sum_{j=0}^{i}\sum_{k,m=0}^{\infty}\Upsilon_{i,j,k,m}(\alpha,\beta)B^*_{\beta k+m}(p), \end{aligned} \quad ...(14)$$

where $B^*_{\beta k+m}(p)=\frac{1}{\mu^* p}\int_0^{G^{-1}(p)} x h_{\beta k+m}(x)dx$ is the Bonferroni curve of the Exp-G family with power parameter $(\beta k+m)$ and μ^* denotes the average. The Lorenz curve can be expressed as

$$L(p) = \sum_{i=1}^{\infty}\sum_{j=0}^{i}\sum_{k,m=0}^{\infty}\Upsilon_{i,j,k,m}(\alpha,\beta)L^*_{\beta k+m}(p), \quad ...(15)$$

where $L^*_{\beta k+m}(p)=\frac{1}{\mu^*}\int_0^{G^{-1}(p)} x h_{\beta k+m}(x)dx$ is the Lorenz curve of the Exp-G family with power parameter $(\beta k+m)$.

Moments of the residual and past lifetimes

For describing different maintenance strategies, we must calculate two important times, namely, the mean residual lifetime (MRL) and the mean past lifetime (MPL). The nth moment of the residual lifetime (RL) is given as

$$M_{RL}^{(n)}(t) = \mathbf{E}((T-t)^n \mid T>t)\ ; n=1,2,3,....$$

Therefore, if $T\sim$ OCh-G family, then

$$\begin{aligned} M_{RL}^{(n)}(t) &= \frac{1}{R(t;\alpha,\beta,\eta)}\int_t^{\infty}(x-t)^n f(x;\alpha,\beta,\eta)dx \\ &= \frac{1}{R(t;\alpha,\beta,\eta)}\sum_{i=1}^{\infty}\sum_{j=0}^{i}\sum_{k,m=0}^{\infty}\Upsilon_{i,j,k,m}(\alpha,\beta) \\ &\quad \int_t^{\infty}(x-t)^n h_{\beta k+m}(x)dx \\ &= \frac{1}{R(t;\alpha,\beta,\eta)}\sum_{i=1}^{\infty}\sum_{j=0}^{i}\sum_{k,m=0}^{\infty}\sum_{l=0}^{n}(-t)^l\binom{n}{l} \\ &\quad \Upsilon_{i,j,k,m}(\alpha,\beta)M_*^{(n)}(t), \end{aligned} \quad ...(16)$$

where $M_*^{(n)}(t)=\int_t^{\infty} x^{n-l}h_{\beta k+m}(x)dx$. Setting n = 1 in equation (16), we get the MRL.

The nth moment of the past lifetime (PL) (also known as the nth moment of the waiting time) can be expressed as

$$\begin{aligned} M_{PL}^{(n)}(t) &= \mathbf{E}((t-T)^n \mid T\le t) \\ &= \frac{1}{F(t;\alpha,\beta,\eta)}\int_0^{t}(t-x)^n f(x;\alpha,\beta,\eta)dx \\ &= \frac{1}{F(t;\alpha,\beta,\eta)}\sum_{i=1}^{\infty}\sum_{j=0}^{i}\sum_{k,m=0}^{\infty}\sum_{l=0}^{n}(-1)^l\, t^{n-l}\binom{n}{l} \\ &\quad \Upsilon_{i,j,k,m}(\alpha,\beta)M_{**}^{(n)}(t), \end{aligned} \quad ...(17)$$

where $M_{**}^{(n)}(t)=\int_0^t x^l h_{\beta k+m}(x)dx;\ n=1,2,3,....$ Setting $n=1$ in equation (17), we get the MPL.

Special OCh-G models

The OCh-Fréchet (OChFr) distribution

Consider the CDF of the Fréchet distribution with positive parameters *a* and *b* given by $G(x;a,b)=e^{-\left(\frac{a}{x}\right)^b},\ x>0$.

Then, the CDF of the OChFr distribution is

$$F(x;\alpha,\beta,a,b) =$$

$$1-\exp\left(-\alpha\left[\exp\left(\exp\left(\frac{a}{x}\right)^{b}-1\right)^{-\beta}-1\right]\right)\ ;\ x>0. \quad ...(18)$$

Figure 1 shows some plots of the PDF and the HRF of the OChFr distribution for various values of the parameters.

Figure 1 shows that the HRF of the OChFr model can be decreasing, increasing or unimodal-shaped. The skewness and kurtosis of the OChFr distribution for some choices of a = 2, b = 2:5 and β as function of α are displayed in Figure 2.

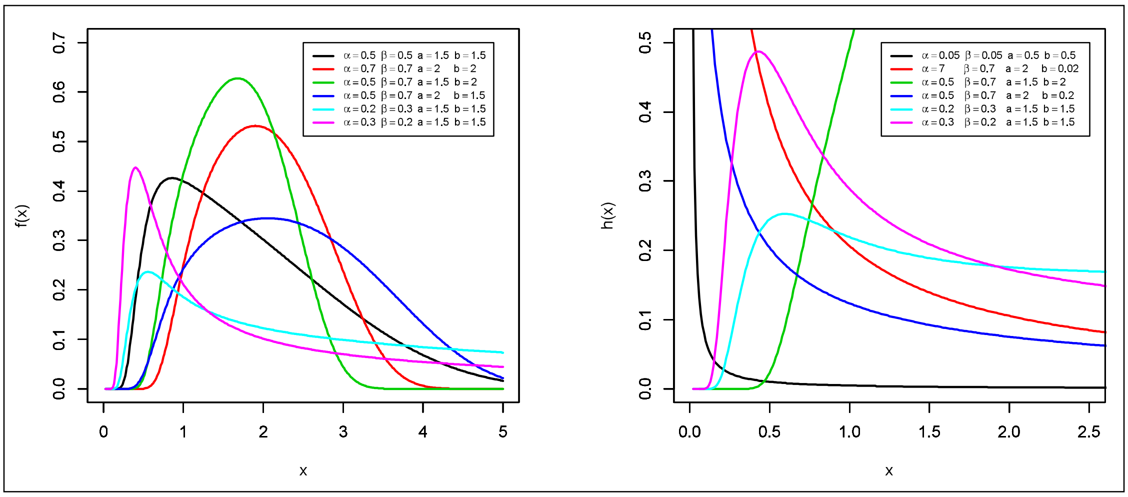

Figure 1: The PDF plots (left panel) and the HRF plots (right panel) of the OChFr distribution

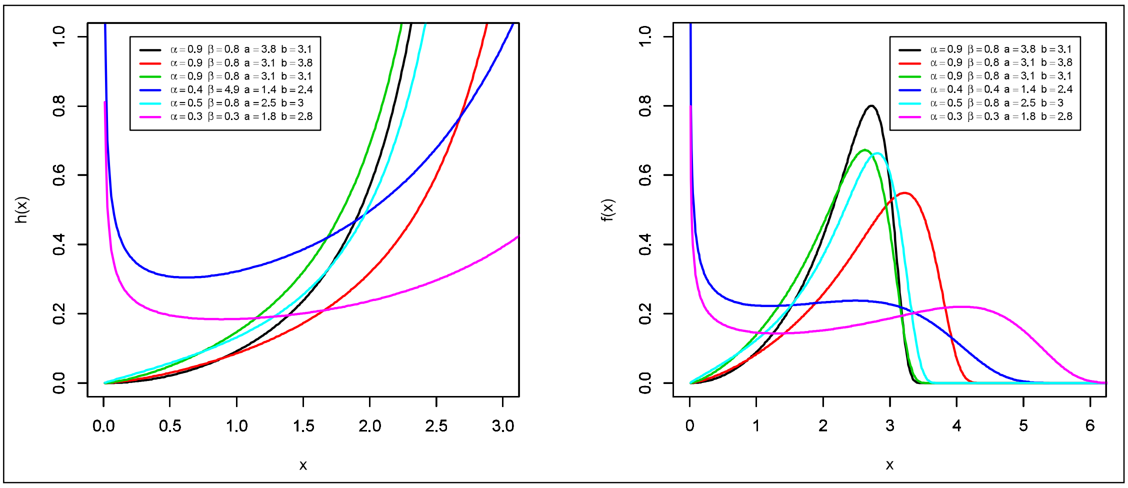

Figure 2: The skewness plots (left panel) and the kurtosis plots (right panel) of the OChFr distribution

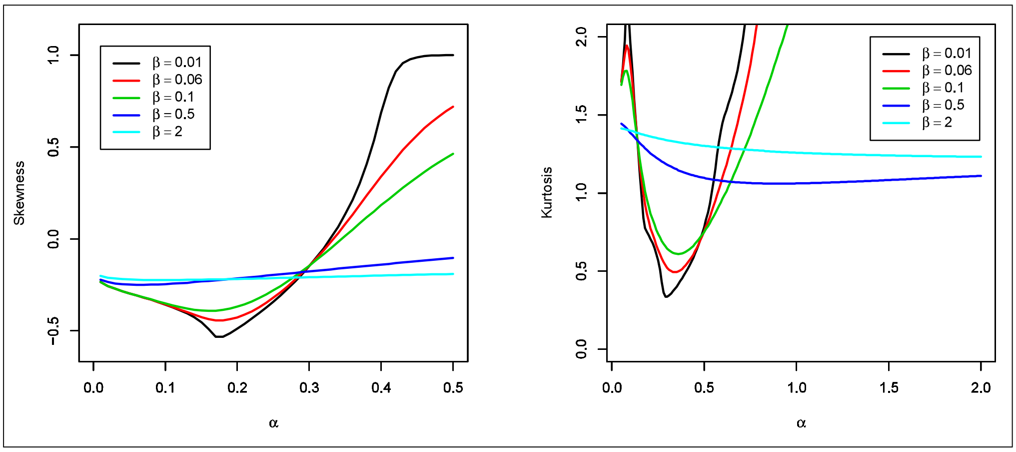

Figure 3: The PDF plots (left panel) and the HRF plots (right panel) of the OChW distribution

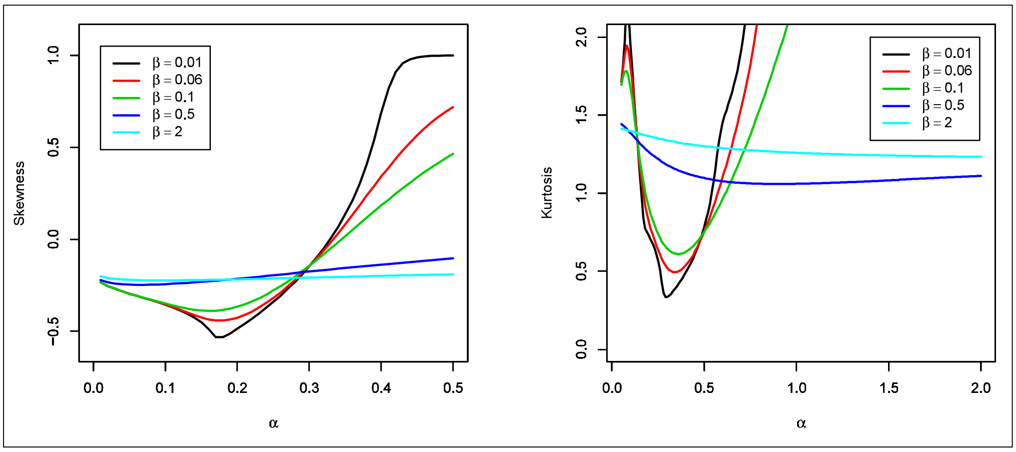

Figure 4: The skewness plots (left panel) and the kurtosis plots (right panel) of the OChW distribution

Figure 2 reveals that the shapes of the OChFr model have strong dependence on the values of α and β.

The OCh-Weibull (OChW) distribution

Consider the CDF of the Weibull distribution with positive parameters a and b given by $G(x;a,b) = 1 - e^{-\left(\frac{x}{b}\right)^a}$, $x > 0$.

Then, the CDF of the OChW distribution is

$$F(x;\alpha,\beta,a,b) = 1-\exp\left[-\alpha\left(\exp\left(\exp\left(\frac{x}{b}\right)^a-1\right)^\beta-1\right)\right],\ x>0. \quad ...(19)$$

Figure 3 shows the PDF and HRF plots of the OChW distribution for various values of the parameters. It can be seen that the HRF can be increasing or bathtub shaped. Further, the skewness and kurtosis of the OChW distribution for some choices of $a = 2$, $b = 0.5$ and β as function of α are displayed in Figure 4.

It is clear from Figure 4, that the shapes of the OChW model have strong dependence on the values of α and β.

Estimation methods

In this section, six methods of estimation are used to estimate the unknown parameters of the OChFr and OChW models to illustrate how different estimators of these distributions perform for various sample sizes and various parameter combinations and to develop a guideline for determining the best method of estimation, which is important for the applied statisticians.

These estimation methods are: the maximum likelihood estimators, least squares estimators, weighted least squares estimators, maximum product of spacings estimators, Cramér-von Mises estimators and percentile based estimators. Similar studies for other models have been proposed by many authors (Eliwa *et al.*, 2018; Nassar *et al.*, 2018; Cordeiro *et al.*, 2019).

Maximum likelihood estimators

Let $X_1, X_2, ..., X_n$ be a random sample of size n from $F(x;\alpha,\beta,a,b)$. Then, the maximum likelihood estimators (MLEs) of the OChFr and OChW parameters $\widehat{\alpha}_{MLE}$, $\widehat{\beta}_{MLE}$, $\widehat{a}_{MLE}$ and $\widehat{b}_{MLE}$ can be obtained by maximizing

$$L(\alpha,\beta,a,b) = \sum_{i=1}^{n} \log f(x_i;\alpha,\beta,a,b), \qquad ...(20)$$

with respect to α,β,a and b.. Or equivalently, the MLEs follow by solving the non-linear equations defined by

$$L_\alpha = \frac{n}{\alpha} - \sum_{i=1}^{n}\left[\left(\exp\left\{\left[\frac{G(x_i;a,b)}{1-G(x;a,b)}\right]^{\beta}\right\} - 1\right)\right] = 0,$$

$$L_\beta = \frac{n}{\beta} + \sum_{i=1}^{n} \ln\left[G(x_i;a,b)\right] - \sum_{i=1}^{n} \ln\left[1-G(x;a,b)\right] +$$

$$\sum_{i=1}^{n}\left\{\frac{G(x_i;a,b)}{[1-G(x;a,b)]}\right\}^{\beta} \ln\left[\frac{G(x_i;a,b)}{1-G(x;a,b)}\right]$$

$$-\alpha\sum_{i=1}^{n} \exp\left\{\left[\frac{G(x_i;a,b)}{1-G(x;a,b)}\right]^{\beta}\right\}\left[\frac{G(x_i;a,b)}{1-G(x;a,b)}\right]^{\beta}$$

$$\ln\left[\frac{G(x_i;a,b)}{1-G(x;a,b)}\right] = 0,$$

$$L_a = \sum_{i=1}^{n}\frac{[g(x_i;a,b)]_a}{g(x_i;a,b)} + (\beta-1)\sum_{i=1}^{n}\frac{[G(x_i;a,b)]_a}{G(x_i;a,b)}$$

$$+ (\beta+1)\sum_{i=1}^{n}\frac{[G(x_i;a,b)]_a}{1-G(x;a,b)}$$

$$+\beta\sum_{i=1}^{n}\left[\frac{[G(x_i;a,b)]^{\beta-1}[G(x_i;a,b)]_a}{[1-G(x;a,b)]^{\beta+1}}\right]$$

$$\left(1-\alpha\exp\left\{\left[\frac{G(x_i;a,b)}{1-G(x;a,b)}\right]^{\beta}\right\}\right) = 0$$

and

$$L_b = \sum_{i=1}^{n}\frac{[g(x_i;a,b)]_b}{g(x_i;a,b)} + (\beta-1)\sum_{i=1}^{n}\frac{[G(x_i;a,b)]_b}{G(x_i;a,b)}$$

$$+ (\beta+1)\sum_{i=1}^{n}\frac{[G(x_i;a,b)]_b}{1-G(x;a,b)} +$$

$$+\beta\sum_{i=1}^{n}\left[\frac{[G(x_i;a,b)]^{\beta-1}[G(x_i;a,b)]_b}{[1-G(x;a,b)]^{\beta+1}}\right]$$

$$\left(1-\alpha\exp\left\{\left[\frac{G(x_i;a,b)}{1-G(x;a,b)}\right]^{\beta}\right\}\right) = 0,$$

where $L_* = \frac{\partial}{\partial *}L(\alpha,\beta,a,b)$ and $[A]_* = \frac{\partial}{\partial *}A$.

Ordinary and weighted least-square estimators

Let $x_{(1)}, x_{(2)}, \cdots, x_{(n)}$ be the order statistics of the random sample of size n from $F(x;\alpha,\beta,a,b)$. The least square estimators (LSEs) of the OChFr and OChW parameters $\widehat{\alpha}_{LS}$, $\widehat{\beta}_{LS}$, $\widehat{a}_{LS}$ and $\widehat{b}_{LS}$ can be obtained by minimizing

$$V(\alpha,\beta,a,b) = \sum_{i=1}^{n}\left[F\left(x_{(i)}|\alpha,\beta,a,b\right) - \frac{i}{n+1}\right]^2,$$

with respect to α,β,a and b.. Or equivalently, the LSEs follow by solving the non-linear equations defined by

$$\sum_{i=1}^{n}\left[F\left(x_{(i)}|\alpha,\beta,a,b\right) - \frac{i}{n+1}\right]\Delta_s\left(x_{(i)}|\alpha,\beta,a,b\right) = 0,$$
$s = 1,2,3,4,$

where

$$\Delta_1\left(x_{(i)}|\alpha,\beta,a,b\right) = \frac{\partial}{\partial\alpha}F\left(x_{(i)}|\alpha,\beta,a,b\right)$$
$$\Delta_2\left(x_{(i)}|\alpha,\beta,a,b\right) = \frac{\partial}{\partial\beta}F\left(x_{(i)}|\alpha,\beta,a,b\right)$$
$$\Delta_3\left(x_{(i)}|\alpha,\beta,a,b\right) = \frac{\partial}{\partial a}F\left(x_{(i)}|\alpha,\beta,a,b\right)$$
$$\Delta_4\left(x_{(i)}|\alpha,\beta,a,b\right) = \frac{\partial}{\partial b}F\left(x_{(i)}|\alpha,\beta,a,b\right) \qquad ...(21)$$

Note that the solution of Δ_s for $s=1,2,3,4$ can be obtained numerically.

The weighted least squares estimators (WLSEs) $\widehat{\alpha}_{WLS}$, $\widehat{\beta}_{WLS}$, $\widehat{a}_{WLS}$ and $\widehat{b}_{WLS}$ can be obtained by minimizing the following equation:

$$W(\alpha,\beta,a,b)=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}$$

$$\left[F\left(x_{(i)}|\alpha,\beta,a,b\right)-\frac{i}{n+1}\right]^2.$$

Further, the WLSEs can also be derived by solving the non-linear equations defined by

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F\left(x_{(i)}|\alpha,\beta,a,b\right)-\frac{i}{n+1}\right]$$

$$\Delta_s\left(x_{(i)}|\alpha,\beta,a,b\right)=0,\quad s=1,2,3,4,$$

where $\Delta_1(\cdot|\alpha,\beta,a,b)$, $\Delta_2(\cdot|\alpha,\beta,a,b)$, $\Delta_3(\cdot|\alpha,\beta,a,b)$ and $\Delta_4(\cdot|\alpha,\beta,a,b)$ are provided in equation (21).

Maximum product of spacings estimators

The maximum product of spacings method is proposed as a good alternative to the MLE method. For $i=1,2,\ldots,n+1$, let

$$D_i(\alpha,\beta,a,b)=F\left(x_{(i)}|\alpha,\beta,a,b\right)-F\left(x_{(i-1)}|\alpha,\beta,a,b\right),$$

be the uniform spacings of a random sample from the OChFr distribution or OChW distribution, where $F\left(x_{(0)}|\alpha,\beta,a,b\right)=0$, $F\left(x_{(n+1)}|\alpha,\beta,a,b\right)=1$ and $\sum_{i=1}^{n+1}D_i(\alpha,\beta,a,b)=1$. The maximum product of spacings estimators (MPSEs) for $\widehat{\alpha}_{MPS}$, $\widehat{\beta}_{MPS}$, $\widehat{a}_{MPS}$ and $\widehat{b}_{MPS}$ can be obtained by maximizing the geometric mean of the spacings

$$G(\alpha,\beta,a,b)=\left[\prod_{i=1}^{n+1}D_i(\alpha,\beta,a,b)\right]^{\frac{1}{n+1}},$$

with respect to α,β,a and b. Or by maximizing the logarithm of the geometric mean of sample spacings

$$H(\alpha,\beta,a,b)=\frac{1}{n+1}\sum_{i=1}^{n+1}\log D_i(\alpha,\beta,a,b).$$

The MPSEs can also be derived by solving the non-linear equations defined by

$$\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{D_i(\alpha,\beta,a,b)}\left[\Delta_s(x_{(i)}|\alpha,\beta,a,b)-\right.$$

$$\left.\Delta_s(x_{(i-1)}|\alpha,\beta,a,b)\right]=0,\ s=1,2,3,4,$$

where $\Delta_1(\cdot|\alpha,\beta,a,b)$, $\Delta_2(\cdot|\alpha,\beta,a,b)$, $\Delta_3(\cdot|\alpha,\beta,a,b)$ and $\Delta_4(\cdot|\alpha,\beta,a,b)$ are defined in equation (21).

Cramér-von Mises minimum distance estimators

Cramér-von Mises estimators (CVMEs) are a type of minimum distance estimators and have less bias than the other minimum distance estimators. The CVMEs are obtained based on the difference between the estimates of the CDF and the empirical distribution function. The CVMEs of the OChFr and OChW parameters are obtained by minimizing

$$C(\alpha,\beta,a,b)=\frac{1}{12n}+\sum_{i=1}^{n}\left[F\left(x_{(i)}|\alpha,\beta,a,b\right)-\frac{2i-1}{2n}\right]^2,$$

with respect to α,β,a and b. Also, the CVMEs follow by solving the non-linear equations

$$\sum_{i=1}^{n}\left[F\left(x_{(i)}|\alpha,\beta,a,b\right)-\frac{2i-1}{2n}\right]\Delta_s\left(x_{(i)}|\alpha,\beta,a,b\right)=0,$$

$$s=1,2,3,$$

where $\Delta_1(\cdot|\alpha,\beta,a,b)$, $\Delta_2(\cdot|\alpha,\beta,a,b)$, $\Delta_3(\cdot|\alpha,\beta,a,b)$ and $\Delta_4(\cdot|\alpha,\beta,a,b)$ are defined in equation (21).

Percentile based estimators

Let $u_i=i/(n+1)$ be an unbiased estimator of $F\left(x_{(i)}|\alpha,\beta,a,b\right)$. Hence, the percentile estimators (PCEs) of the OChFr and OChW parameters can be obtained by minimizing

$$P(\alpha,\beta,a,b)=\sum_{i=1}^{n}\left(x_{(i)}-Q(u_i)\right)^2,$$

with respect to α,β,a and b, where $Q(u_i)$ is the quantile function of the OChFr and OChW distributions which are given, respectively, by using equation (9).

Table 1: The AEs with their corresponding MSEs (in parentheses) for the OChFr model

n	Par	MLEs	LSEs	WLSEs	MPSEs	CVMEs	PCEs
50	α	0.688 (0.201)	0.691 (0.215)	0.733 (0.274)	0.703 (0.265)	0.701 (0.259)	0.736 (0.283)
	β	1.714 (0.304)	1.687 (0.245)	1.726 (0.315)	1.701 (0.289)	1.705 (0.294)	1.729 (0.337)
	a	2.234 (0.268)	2.247 (0.281)	2.257 (0.288)	2.289 (0.292)	2.251 (0.284)	2.239 (0.276)
	b	2.604 (0.216)	2.598 (0.187)	2.631 (0.230)	2.618 (0.220)	2.601 (0.212)	2.644 (0.259)
150	α	0.602 (0.146)	0.623 (0.201)	0.701 (0.265)	0.687 (0.254)	0.655 (0.209)	0.700 (0.223)
	β	1.656 (0.267)	1.642 (0.226)	1.687 (0.294)	1.640 (0.251)	1.604 (0.197)	1.645 (0.249)
	a	2.187 (0.222)	2.202 (0.234)	2.200 (0.218)	2.218 (0.241)	2.203 (0.226)	2.213 (0.228)
	b	2.587 (0.186)	2.567 (0.129)	2.610 (0.214)	2.600 (0.208)	2.548 (0.179)	2.601 (0.201)
300	α	0.546 (0.087)	0.611 (0.187)	0.643 (0.200)	0.546 (0.201)	0.601 (0.088)	0.589 (0.115)
	β	1.600 (0.147)	1.587 (0.147)	1.601 (0.215)	1.576 (0.157)	1.533 (0.127)	1.609 (0.193)
	a	2.112 (0.143)	2.102 (0.098)	2.174 (0.116)	2.179 (0.200)	2.132 (0.163)	2.178 (0.107)
	b	2.514(0.069)	2.513 (0.091)	2.534 (0.097)	2.543 (0.146)	2.508 (0.009)	2.574 (0.135)
500	α	0.501 (0.002)	0.523 (0.041)	0.537 (0.074)	0.515 (0.042)	0.516 (0.007)	0.511 (0.010)
	β	1.522 (0.010)	1.516 (0.001)	1.509 (0.012)	1.510 (0.010)	1.510 (0.030)	1.513 (0.019)
	a	2.011 (0.002)	2.015 (0.022)	2.008 (0.001)	2.012 (0.024)	2.074 (0.077)	2.010 (0.004)
	b	2.504 (0.001)	2.505 (0.001)	2.503 (0.001)	2.508 (0.002)	2.500 (0.000)	2.507 (0.001)

Table 2: The AEs with their corresponding MSEs (in parentheses) for the OChW model

n	Par	MLEs	LSEs	WLSEs	MPSEs	CVMEs	PCEs
50	α	2.717 (0.225)	2.747 (0.276)	2.713 (0.256)	2.765 (0.273)	2.777 (0.247)	2.701 (0.244)
	β	2.243 (0.284)	2.215 (0.266)	2.288 (0.263)	2.226 (0.274)	2.192 (0.212)	2.214 (0.253)
	a	1.678 (0.190)	1.611 (0.149)	1.674 (0.141)	1.626 (0.188)	1.610 (0.173)	1.647 (0.122)
	b	0.598 (0.115)	0.599 (0.193)	0.579 (0.180)	0.584 (0.112)	0.571 (0.109)	0.576 (0.160)
150	α	2.701 (0.219)	2.675 (0.223)	2.647 (0.241)	2.742 (0.258)	2.724 (0.222)	2.650 (0.213)
	β	2.201 (0.224)	2.196 (0.134)	2.234 (0.234)	2.201 (0.233)	2.172 (0.201)	2.201 (0.226)
	a	1.626 (0.121)	1.584 (0.101)	1.622 (0.130)	1.601 (0.154)	1.587 (0.143)	1.618 (0.111)
	b	0.572 (0.089)	0.534 (0.082)	0.564 (0.110)	0.561 (0.089)	0.562 (0.058)	0.556 (0.070)
300	α	2.566 (0.159)	2.632 (0.187)	2.601 (0.189)	2.684 (0.213)	2.675 (0.200)	2.631 (0.202)
	β	2.125 (0.148)	2.142 (0.109)	2.173 (0.174)	2.155 (0.202)	2.142 (0.176)	2.167 (0.205)
	a	1.600 (0.077)	1.552 (0.074)	1.600 (0.100)	1.574 (0.123)	1.555 (0.116)	1.606 (0.107)
	b	0.542 (0.021)	0.521 (0.020)	0.556 (0.074)	0.555 (0.071)	0.531 (0.023)	0.549 (0.061)
500	α	2.505 (0.002)	2.523 (0.007)	2.525 (0.055)	2.534 (0.088)	2.520 (0.081)	2.531 (0.112)
	β	2.014 (0.003)	2.045 (0.031)	2.110 (0.056)	2.079 (0.097)	2.077 (0.086)	2.120 (0.099)
	a	1.511 (0.008)	1.506 (0.003)	1.533 (0.037)	1.522 (0.029)	1.507 (0.005)	1.534(0.074)
	b	0.500 (0.000)	0.504 (0.001)	0.514 (0.005)	0.506 (0.002)	0.501 (0.000)	0.518 (0.007)

Simulation

In this section, a simulation study is conducted to compare the performance of the MLEs, LSEs, WLSEs, MPSEs, CVMEs and PCEs of the unknown parameters for the OChFr and OChW distributions with respect to sample size n: Mathcad V.15 is used herein to generate the samples as follows:

1. Generate 10000 samples of size n = 50, 150, 300, 500 from OChFr (α, β, a, b) = (0.5, 1.5, 2.0, 2.5) and OChW(α, β, a, b) = (2.5, 2.0, 1.5, 0.5), respectively.
2. Compute the MLEs, LSEs, WLSEs, MPSEs, CVMEs and PCEs for the 10,000 samples, say $\widehat{\alpha}_j, \widehat{\beta}_j, \widehat{a}_j$ and $\widehat{b}_j$ for j = 1, 2, ..., 10,000.
3. Compute the average values of estimates (AEs) and mean-squared errors (MSEs).

The empirical results are given in Tables 1 and 2, respectively.

Regarding Tables 1 and 2, the following observations can be made:

1. The magnitude of bias always decreases to zero as $n \to \infty$.
2. The MSEs decrease when the sample size increases as expected under first-order asymptotic theory.
3. Depending on the MSEs, the MLE, LSE, WLSE, MPSE, CVME and PUCE methods perform quite well for estimating the OChFr and OChW parameters. However, we can consider the MLE, WLSE and CVME methods outperform LSE, MPSE and PCE methods. Therefore, the MLE, WLSE and CVME method are the best.

RESULTS AND DISCUSSION: DATA ANALYSIS

In this section, we illustrate the empirical importance of the OChFr and OChW distributions using two applications to real data.

The first dataset (I): represents the survival times, in weeks, of 33 patients suffering from acute Myelogeneous Leukaemia (Feigl & Zelen, 1965). For the dataset I, we shall compare the fits of the OChFr distribution with some competitive models listed in Table 3.

The second dataset (II): represents 40 observations of time-to-failure (10^3h) of turbocharger of one type of engine (Xu *et al*., 2003). This dataset is used to compare the fits of the OChW model with some competitive models provided in Table 4.

The fitted distributions are compared using some criteria, namely, the maximized log likelihood ($-2L$), Akaike information criterion (AIC), Cramér-von Mises (W^*) statistic, Anderson-Darling (A^*) statistic, Kolmogorov-Smirnov (KS) statistic and its p value. Tables 5 and 7 list the MLEs with their corresponding standard errors (SEs) (in parentheses) for both datasets, respectively, whereas Tables 6 and 8 provide the values of goodness-of-fit measures for datasets I and II, respectively.

Table 3: The competitive models of the OChFr distribution for dataset I

Distribution	Abbreviation	Author(s)
Weibull Fr	WFr	Afify *et al*. (2016b)
Kumaraswamy Marshall-Olkin Fr	KMOFr	Afify *et al*. (2016a)
Kumaraswamy Fr	KFr	Mead and Abd-Eltawab (2014)
Exponentiated Fr	EFr	Nadarajah and Kotz (2003)
Transmuted exponentiated Fr	TEFr	Elbatal *et al*. (2014)
Beta exponential Fr	BExFr	Mead *et al*. (2017)
Fr	Fr	Frechet (1924)

Table 4: The competitive models of the OChW distribution for dataset II

Distribution	Abbreviation	Author(s)
Odd log-logistic exponentiated W	OLLEW	Afify *et al*. (2018)
Alpha logarithmic transformed W	ALTW	Nassar *et al*. (2018)
Beta W	BW	Lee *et al*. (2007)
Marshall-Olkin W	MOW	Ghitany *et al*. (2005)
Transmuted complementary W geometric	TCWG	Afify *et al*. (2014)
Lindley W	LiW	Cordeiro *et al*. (2018)
W	W	Weibull (1951)

The values in Tables 6 and 8 show that the OChFr and OChW distributions have the lowest values of $-2L$, AIC, W^*, A^* and KS measures and then provide the best fits to both datasets. Furthermore, the p value test for the OChFr and OChW models have the largest value among all models. Hence, the OChFr and OChW distributions yield a better fit to those datasets than other distributions.

Figures 5 – 10 show the fitted PDFs, estimated CDFs and P-P plots of the tested distributions for both data sets.

Table 5: The MLEs with their SEs for dataset I

Model	MLEs	SEs
OChFr	$\hat{\alpha} = 0.197$, $\hat{\beta} = 0.210$, $\hat{a} = 5.468$, $\hat{b} = 1.490$	(0.099), (0.092), (2.803), (0.607)
WFr	$\hat{\alpha} = 2.622$, $\hat{\beta} = 1.838$, $\hat{a} = 0.170$, $\hat{b} = 0.380$	(1.835), (1.465), (0.145), (0.271)
KMOFr	$\hat{\alpha} = 31946.7$, $\hat{\beta} = 0.607$, $\hat{\delta} = 4.806$, $\hat{a} = 1.014$, $\hat{b} = 13724.5$	(57.93), (0.106), (6.101), (0.139), (106.87)
KFr	$\hat{\alpha} = 9378.5$, $\hat{\beta} = 0.084$, $\hat{a} = 5.513$, $\hat{b} = 7160.5$	(80.43), (0.024), (17.49), (36.07)
EFr	$\hat{\alpha} = 1426.6$, $\hat{\beta} = 0.249$, $\hat{\delta} = 13.746$	(36.07), (0.070), (13.512)
TEFr	$\hat{\alpha} = 1250.6$, $\hat{\beta} = 0.249$, $\hat{a} = 13.487$, $\hat{b} = -0.117$	(33.903), (0.076), (13.871), (0.487)
BExFr	$\hat{\alpha} = 0.116$, $\hat{\beta} = 4.364$, $\hat{\delta} = 0.043$, $\hat{a} = 9.384$, $\hat{b} = 6.405$	(0.026), (0.025), (0.048), (2.709), (9.696)
Fr	$\hat{a} = 7.865$, $\hat{b} = 0.694$	(2.091), (0.091)

Table 6: Goodness-of-fit statistics for dataset I

Model	–2L	AIC	W^*	A^*	KS	p-value
OChFr	300.183	308.183	0.0417	0.3148	0.105	0.863
WFr	302.577	310.577	0.0613	0.4255	0.114	0.780
KMOFr	304.804	314.804	0.0836	0.5469	0.140	0.533
KFr	304.832	314.832	0.0946	0.6342	0.139	0.546
EFr	307.788	313.788	0.1115	0.7050	0.135	0.581
TEFr	307.760	315.760	0.1104	0.7006	0.136	0.569
BExFr	309.905	319.905	0.1393	0.8549	0.140	0.536
Fr	311.997	315.997	0.1601	0.9759	0.149	0.456

Table 7: The MLEs with their SEs for dataset II

Model	MLEs	SEs
OChW	$\hat{\alpha} = 0.928$, $\hat{\beta} = 0.164$, $\hat{a} = 15.473$, $\hat{b} = 8.273$	(0.254), (0.032), (0.582), (0.229)
OLLEW	$\hat{\alpha} = 8.309$, $\hat{\beta} = 17.125$, $\hat{\gamma} = 0.188$, $\hat{\theta} = 0.771$	(0.148), (0.157), (0.046), (0.153)
ALTW	$\hat{\alpha} = 20001.4$, $\hat{\beta} = 2.654$, $\hat{\lambda} = 0.032$	(105.96), (0.409), (0.028)
BW	$\hat{\alpha} = 0.075$, $\hat{\beta} = 11.242$, $\hat{a} = 0.240$, $\hat{b} = 115.43$	(0.030), (3.850), (0.102), (48.90)
MOW	$\hat{\alpha} = 0.188$, $\hat{\beta} = 2.787$, $\hat{\lambda} = 4.857$	(0:046), (0:873), (5:668)
TCWG	$\hat{\alpha} = 0.2059$, $\hat{\beta} = 2.7881$, $\hat{\lambda} = -8.9{\cdot}10^{-5}$, $\hat{\sigma} = 0.188$	(0:2747), (0:8733), (0:647), (0:046)
LiW	$\hat{\alpha} = 0.169$, $\hat{\beta} = 3.499$, $\hat{\theta} = 0.898$	(0:073), (0:633), (1:093)
W	$\hat{a} = 3.872$, $\hat{b} = 6.920$	(0:517), (0:294)

Table 8: Goodness-of-fit statistics for dataset II

Model	–2*L*	AIC	W	A	KS	p value
OChW	155.058	163.058	0.0111	0.0857	0.046	0.999
OLLEW	155.801	163.801	0.0159	0.1018	0.062	0.998
ALTW	157.218	163.218	0.0154	0.1221	0.059	0.998
BW	158.076	166.076	0.0210	0.1696	0.088	0.917
MOW	162.640	168.64	0.0496	0.3766	0.092	0.889
TCWG	162.640	170.640	0.0496	0.3766	0.092	0.889
LiW	163.775	169.775	0.0636	0.4815	0.102	0.802
W	164.951	168.951	0.0769	0.5730	0.108	0.742

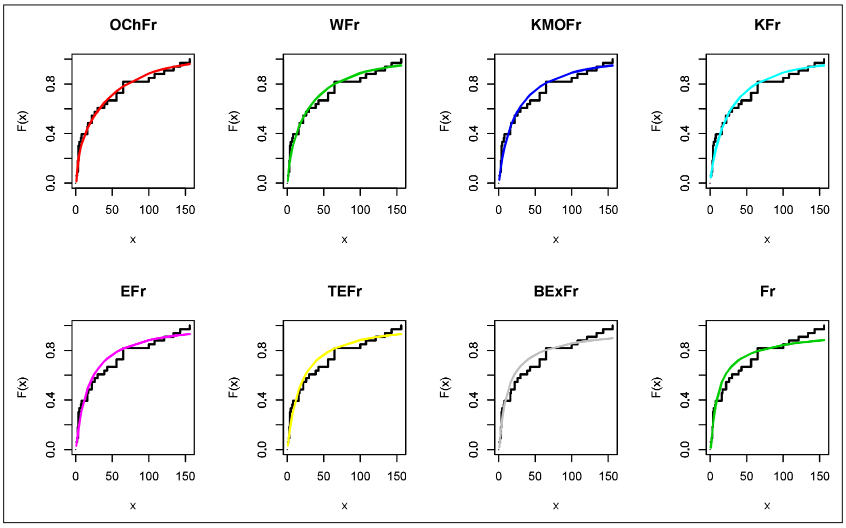

Figure 5: The estimated CDFs for dataset I

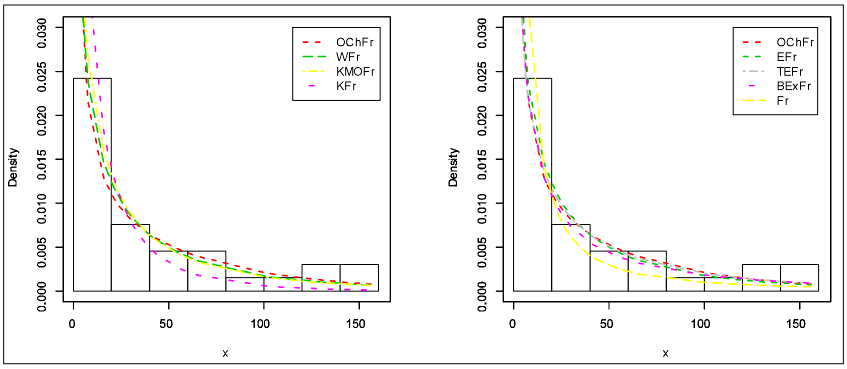

Figure 6: The fitted PDFs for dataset I

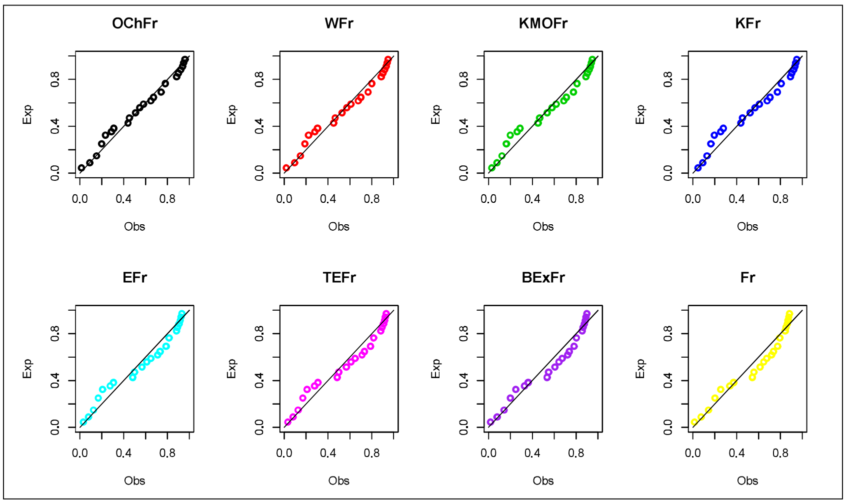

Figure 7: The P-P plots for dataset I

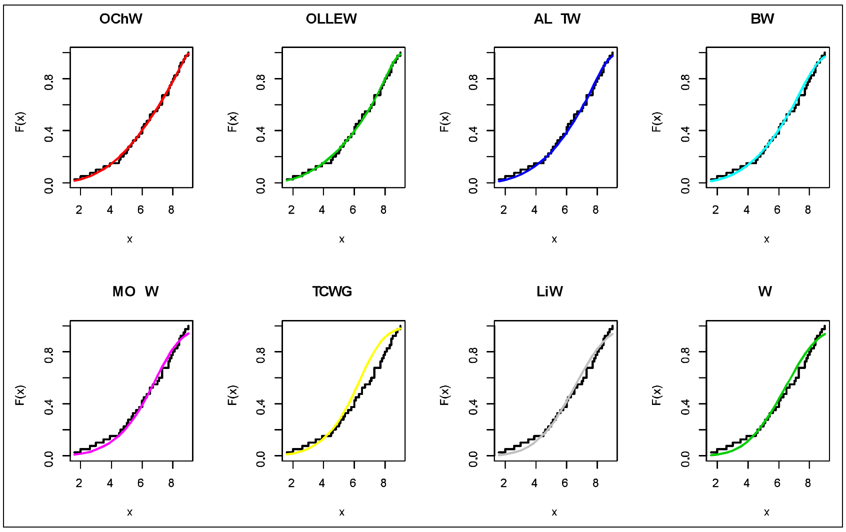

Figure 8: The estimated CDFs for dataset II

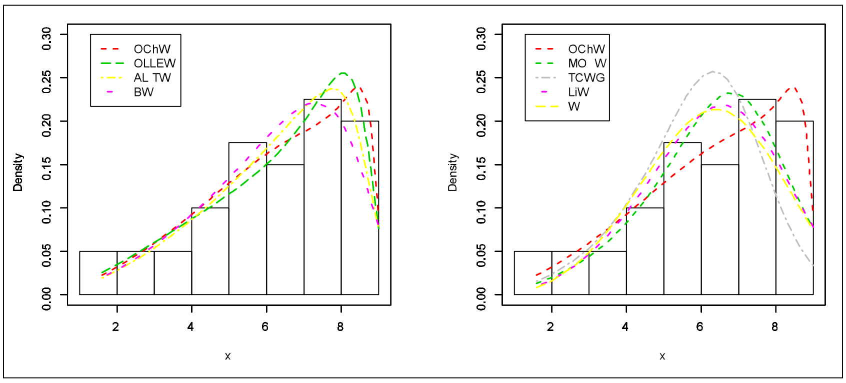

Figure 9: The fitted PDFs for dataset II

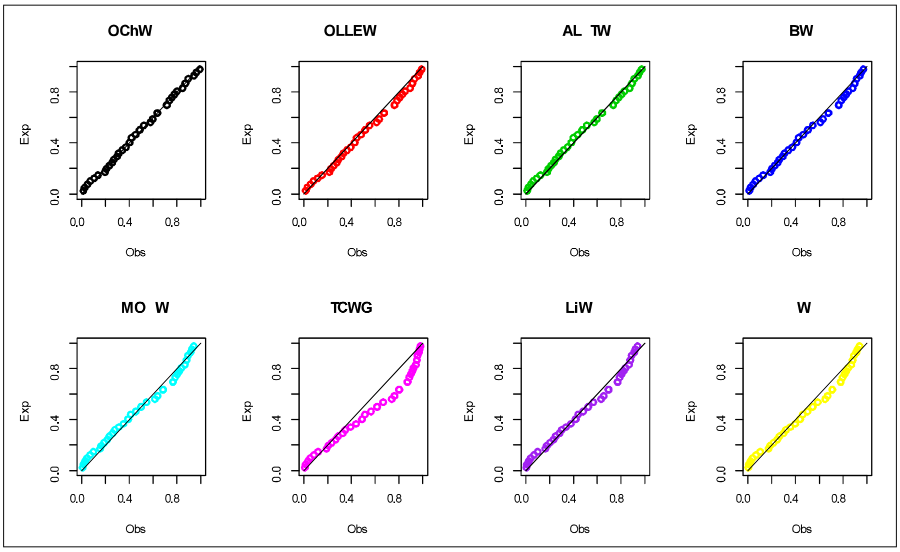

Figure 10: The P-P plots for dataset II

Table 9: Various estimators of the OChFr parameters, KS and p-values for data set I

Method ↓ Estimator →	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{a}$	$\widehat{b}$	KS	p value
LSE	0.309	1.266	2.518	0.212	0.116	0.770
WLSE	0.459	1.513	4.088	0.195	0.107	0.848
MPSE	0.284	0.362	8.091	0.874	0.117	0.754
CVME	0.157	0.873	1.259	0.288	0.113	0.791
PCE	0.142	0.153	4.092	2.046	0.114	0.783

Table 10: Various estimators of the OChW parameters, KS and p-values for dataset II

Method ↓ Estimator →	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{a}$	$\widehat{b}$	KS	p value
LSE	0.353	0.816	1.879	7.749	0.057	0.998
WLSE	0.244	0.450	2.609	6.503	0.073	0.983
MPSE	0.240	0.410	2.501	6.407	0.104	0.775
CVME	0.007	0.118	0.544	0.059	0.057	0.998
PCE	0.358	0.817	1.964	7.749	0.053	0.997

Now, the different methods of estimation mentioned previously will be used to estimate the unknown parameters of the OChFr and OChW models. The KS statistic and its p value are provided to verify the best estimators. Tables 9 and 10 report the estimates of the unknown parameters using five estimation methods and the values of KS with corresponding p value for datasets I and II, respectively.

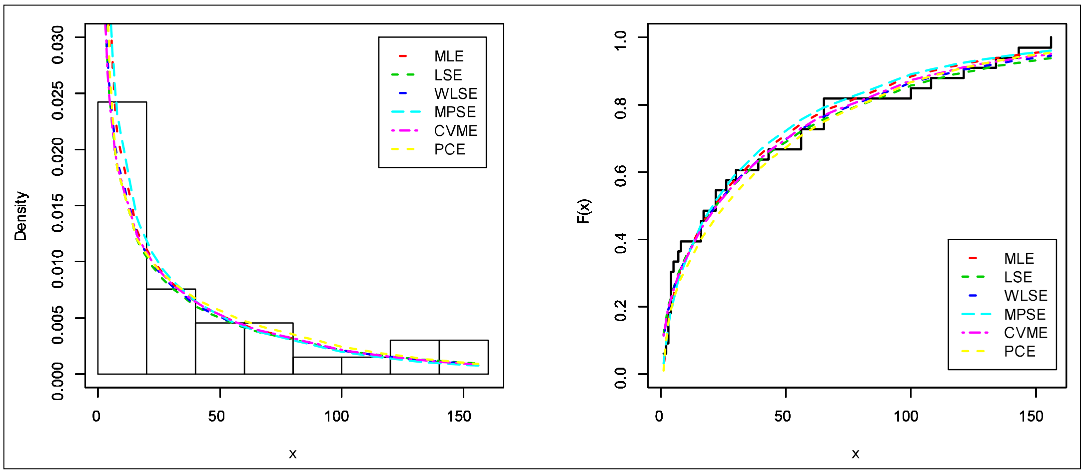

Figure 11: The fitted densities (left panel) and estimated CDFs (right panel) of the OChFr distribution for various estimation methods for dataset I

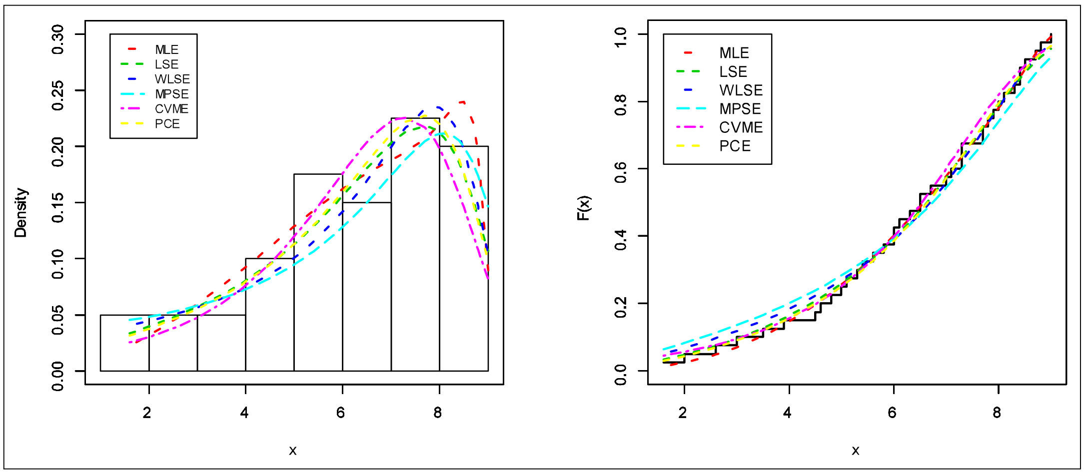

Figure 12: The fitted densities (left panel) and estimated CDFs (right panel) of the OChW distribution for various estimation methods for dataset II

Tables 9 and 10 illustrate that all estimation methods work quite well. However, the MLE method gives the best estimation for the model parameters and consequently we recommend using it for estimating the model parameters for both datasets. Figures 11, 12 and 13 show the fitted PDFs, estimated CDFs and P-P plots for both datasets using the estimators in Tables 9 and 10.

Tables 11 and 12 list some numerical values of some reliability concepts for datasets I and II.

It is seen, from Table 11, that the RF and HRF decrease and the MRL increases with $t \to \infty$, whereas the RF and MRL decrease and the HRF increases with $t \to \infty$ as seen from Table 12.

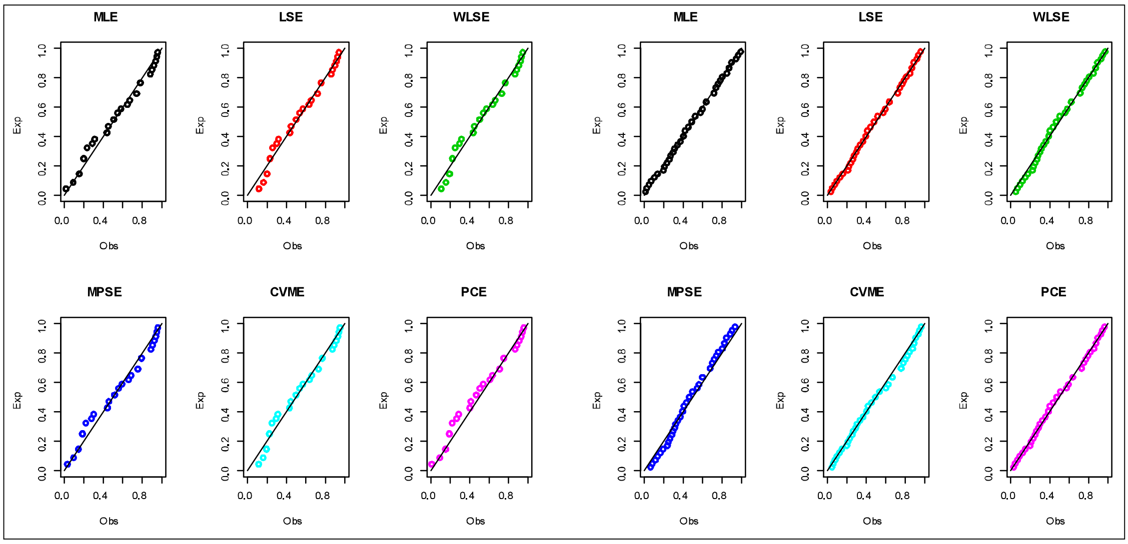

Figure 13: The P-P plots for various estimation methods for datasets I (left panel) and II (right panel)

Table 11: The RF, HRF and MRL for dataset I

Measure ↓ Time →	3	6	9	12	15	18	21	24	27
RF	0.847	0.738	0.669	0.617	0.574	0.536	0.503	0.472	0.445
HRF	0.062	0.037	0.028	0.025	0.023	0.022	0.021	0.020	0.019
MRL	42.172	45.237	46.679	47.499	47.989	48.269	48.401	48.423	48.356

Table 12: The RF, HRF and MRL for dataset II

Method ↓ Estimator →	2	3	4	5	6	7	8	9
RF	0.975	0.929	0.853	0.742	0.596	0.421	0.221	0.008
HRF	0.032	0.064	0.109	0.173	0.272	0.444	0.993	11.121
MRL	4.263	3.445	2.704	2.029	1.399	0.768	0.667	0.364

CONCLUSIONS

In this study, we proposed a new generator of distributions called the odd Chen-G (OCh-G) family. Several of its statistical properties have been derived. The special sub models of the OCh-G family are capable of modelling symmetric and positive as well as negative skewness datasets. Moreover, these sub models provide a wide variation in the shape of the hazard rate, including decreasing, increasing, unimodal and bathtub shapes, and consequently the generated model can be used in modelling various types of data. Two special cases of the OCh-G family, called the OCh-Frechet and OCh-Weibull models were studied. The model parameters are estimated using six different estimation methods, namely, the maximum likelihood estimators, least squares estimators, weighted least squares estimators, maximum product of spacings estimators, Cramér-von

Mises estimators and percentile based estimators. The maximum likelihood estimation gives the best estimators for both OCh-Frechet and OCh-Weibull models. Finally, the two special cases of the OCh-G class are applied to two real datasets from the medicine and engineering fields to illustrate the flexibility of the proposed family.

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