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RESEARCH ARTICLE

Utilising bivariate auxiliary information for enhanced estimation of population mean under simple and stratified random sampling schemes

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Abstract: The present work suggests some difference-cum-exponential ratio-type estimators to deal with the problem of estimation for population mean. The suggested estimators are based on the linear combination of two auxiliary variables under simple and stratified random sampling schemes. Expressions for the bias, mean squared error (MSE) and minimum MSE of the suggested estimators are derived up to the first degree of approximation. Different real life datasets are used to show the superiorities in terms of percent relative efficiencies (PREs) of the new estimators. The suggested estimators are more efficient as they provide maximum gain in PREs as compared to the traditional and competing estimators under study.

Keywords: Auxiliary variable, bias, mean squared error, percent relative efficiency, stratified random sampling.

INTRODUCTION

In sample surveys, information on auxiliary variate(s) has an essential role in enhancing the efficiency of estimators of the population parameters. Considerable work has been done for the estimation of population mean by utilising bivariate auxiliary information under simple and stratified random sampling schemes. Moving along this direction, some noteworthy contributions in simple random sampling without replacement (SRSWOR) have been developed by various authors including Olkin (1958), Abu-Dayyeh *et al*. (2003), Kadilar and Cingi (2005), Singh and Tailor (2005), Perri (2007), Lu (2013), Lu and Yan (2014), Lu *et al*. (2014), Subramani and Prabavathy (2014), Sharma and Singh (2014, 2015), Vishwakarma and Kumar (2015), Muneer *et al*. (2017), Shabbir and Gupta (2017) and many others. Similar efforts have been carried out in stratified random sampling by Koyuncu and Kadilar (2009), Tailor *et al*. (2012), Tailor and Chouhan (2014), Lone *et al*. (2016; 2017), Muneer *et al*. (2017) and Shabbir and Gupta (2017).

The objective of this study is to increase the precision of difference-cum-exponential ratio-type estimators through the linear combination of two auxiliary variables in simple and stratified random sampling.

Existing estimators in simple random sampling

Consider a sample of size n drawn by SRSWOR from a finite population $\Theta = \{\Theta_1, \Theta_2, \Theta_3, \dots, \Theta_N\}$ of size N with $n < N$. Let y_i and (x_i, z_i) denote the observations on the study variable (y) and the auxiliary variables (x, z), respectively for the i^{th} unit of the population. Some useful measures related to the variables under study are given below.

Let the sample means be,

$$\bar{y} = n^{-1}\sum_{i=1}^{n} y_i, \quad \bar{x} = n^{-1}\sum_{i=1}^{n} x_i, \quad \bar{z} = n^{-1}\sum_{i=1}^{n} z_i$$

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The corresponding population means are,

$$\bar{Y} = N^{-1}\sum_{i=1}^{N} y_i, \quad \bar{X} = N^{-1}\sum_{i=1}^{N} x_i, \quad \bar{Z} = N^{-1}\sum_{i=1}^{N} z_i$$

Following are the population parameters represented by their respective subscripts:

The standard deviations are,

$$S_y = \sqrt{(N-1)^{-1}\sum_{i=1}^{N}(y_i - \bar{Y})^2}$$

$$S_x = \sqrt{(N-1)^{-1}\sum_{i=1}^{N}(x_i - \bar{X})^2}$$

$$S_z = \sqrt{(N-1)^{-1}\sum_{i=1}^{N}(z_i - \bar{Z})^2}$$

The covariances are,

$$S_{yx} = (N-1)^{-1}\sum_{i=1}^{N}(y_i - \bar{Y})(x_i - \bar{X})$$

$$S_{xz} = (N-1)^{-1}\sum_{i=1}^{N}(x_i - \bar{X})(z_i - \bar{Z})$$

$$S_{yz} = (N-1)^{-1}\sum_{i=1}^{N}(y_i - \bar{Y})(z_i - \bar{Z})$$

The coefficients of determination are,

$C_y^2 = (\bar{Y}^2)^{-1}S_y^2,\ C_x^2 = (\bar{X}^2)^{-1}S_x^2,\ C_z^2 = (\bar{Z}^2)^{-1}S_z^2$

The correlation coefficients are $\rho_{yx} = \left(S_y S_x\right)^{-1} S_{yx}$, $\rho_{xz} = (S_x S_z)^{-1} S_{xz}$, $\rho_{yz} = \left(S_y S_z\right)^{-1} S_{yz}$

The finite population correction factor is $\psi = \left(\frac{1}{n} - \frac{1}{N}\right)$

Let $C_{yx} = \rho_{yx} C_y C_x,\ C_{yz} = \rho_{yz} C_y C_z,\ C_{xz} = \rho_{xz} C_x C_z$

$R_1 = \frac{\bar{Y}}{\bar{X}}, \quad R_2 = \frac{\bar{X}}{\bar{Y}}, \quad R_3 = \frac{\bar{Z}}{\bar{Y}}, \quad R_4 = \frac{\bar{Y}}{\bar{Z}}$ and

$$R_{y.xz}^2 = \frac{\rho_{yx}^2 + \rho_{yz}^2 - 2\rho_{yx}\rho_{yz}\rho_{xz}}{1 - \rho_{xz}^2}$$

A brief introduction of some well-known estimators based on the information of two auxiliary variables for the estimation of population mean from literature are given below.

(1) Commonly used unbiased estimator of population mean $\bar{Y}$ is,

$$\hat{\bar{Y}}_O = n^{-1}\sum_{i=1}^{n} y_i \qquad ...(1)$$

The variance of $\hat{\bar{Y}}_O$ is given by,

$$V\left(\hat{\bar{Y}}_O\right) = \psi \bar{Y}^2 C_y^2 = MSE\left(\hat{\bar{Y}}_O\right) \qquad ...(2)$$

(2) Traditional multivariate ratio type estimator suggested by Olkin (1958) is given by,

$$\hat{\bar{Y}}_{MR} = \bar{y}\left(\omega_1 \frac{\bar{X}}{\bar{x}} + \omega_2 \frac{\bar{Z}}{\bar{z}}\right) \qquad ...(3)$$

where ω_1 and ω_2 are the unknown constants under the condition $\omega_1 + \omega_2 = 1$. The optimum values of ω_1 and ω_2 are determined as,

$$\omega_{1(opt)} = \frac{C_z^2 + C_{yx} - C_{yz} - C_{xz}}{C_x^2 + C_z^2 - 2C_{xz}} \text{ and } \omega_{2(opt)} = 1 - \omega_{1(opt)}$$

With the help of the above optimum values of ω_1 and ω_2, we obtained the minimum MSE of $\hat{\bar{Y}}_{MR}$ to the first degree of approximation as follows;

$$MSE_{min}\left(\hat{\bar{Y}}_{MR}\right) \cong \psi\bar{Y}^2\left(C_y^2 + C_z^2 - 2C_{yz} - \frac{\left(C_z^2 + C_{yx} - C_{yz} - C_{xz}\right)^2}{C_x^2 + C_z^2 - 2C_{xz}}\right) \qquad ...(4)$$

(3) The traditional multivariate regression estimator $\hat{\bar{Y}}_{MReg}$ is given as,

$$\hat{\bar{Y}}_{MReg} = \bar{y} + b_{yx}(\bar{X} - \bar{x}) + b_{yz}(\bar{Z} - \bar{z}) \qquad ...(5)$$

where $b_{yx} = (S_x^2)^{-1} S_{yx}$ and $b_{yz} = (S_z^2)^{-1} S_{yz}$ are the sample regression coefficients.

The MSE of the estimator $\hat{\bar{Y}}_{MReg}$ to the first order of approximation is given by,

$$MSE\left(\hat{\bar{Y}}_{MReg}\right) \cong \psi\bar{Y}^2 C_y^2\left(1 - \rho_{yx}^2 - \rho_{yz}^2 + 2\rho_{yx}\rho_{yz}\rho_{xz}\right) \qquad ...(6)$$

(4) The traditional difference estimator is given by,

$$\hat{\bar{Y}}_D = \bar{y} + \omega_3(\bar{X} - \bar{x}) + \omega_4(\bar{Z} - \bar{z}) \qquad ...(7)$$

where ω_3 and ω_4 are the suitably chosen weights.

Minimum MSE of $\hat{\hat{Y}}_D$ at optimum values of $\omega_{3(opt)} = R_1\left[\frac{C_y(\rho_{yx}-\rho_{yz}\rho_{xz})}{C_x(1-\rho_{xz}^2)}\right]$ and $\omega_{4(opt)} = R_4\left[\frac{C_y(\rho_{yz}-\rho_{yx}\rho_{xz})}{C_z(1-\rho_{xz}^2)}\right]$ is given below;

$$MSE_{min}\left(\hat{\hat{Y}}_D\right) \cong \psi\bar{Y}^2 C_y^2\left(1 - R_{y.xz}^2\right) \qquad \ldots(8)$$

(5) On the lines of Gupta and Shabbir (2008) and Singh and Singh (2014), Muneer *et al*. (2017) proposed the following estimators;

$$\hat{\hat{Y}}_{MU,\gamma} = [\omega_5\bar{y} + \omega_6(\bar{X} - \bar{x})]\left[\gamma\left\{2 - \exp\left(\frac{\bar{z} - \bar{Z}}{\bar{z} + \bar{Z}}\right)\right\} + (1-\gamma)\exp\left(\frac{\bar{Z} - \bar{z}}{\bar{Z} + \bar{z}}\right)\right] \qquad \ldots(9)$$

where ω_5 and ω_6 are the constants to be determined. By using $\gamma = 1$ and 0, we obtained two different estimators $\hat{\hat{Y}}_{MU,1}$ and $\hat{\hat{Y}}_{MU,0}$, respectively. The optimal values of ω_5 and ω_6 are obtained as follows;

$$\omega_{5(Opt)} = \frac{1 + \left(\frac{3}{8} - \frac{\gamma}{4}\right)\psi C_z^2 - \frac{1}{2}\psi C_{yz} - \frac{\psi C_{xz}\left(C_{xz} - C_{yx}\right)}{2C_x^2}}{1 + \psi C_y^2 + \left(1 - \frac{\gamma}{2}\right)\psi C_z^2 - 2\psi C_{yz} - \frac{\psi\left(C_{xz} - C_{yx}\right)^2}{C_x^2}}$$

$$\omega_{6(Opt)} = R_1\left[\frac{C_{xz}}{2C_x^2} - \frac{1 + \left(\frac{3}{8} - \frac{\gamma}{4}\right)\psi C_z^2 - \frac{1}{2}\psi C_{yz} - \frac{\psi C_{xz}\left(C_{xz} - C_{yx}\right)}{2C_x^2}}{1 + \psi C_y^2 + \left(1 - \frac{\gamma}{2}\right)\psi C_z^2 - 2\psi C_{yz} - \frac{\psi\left(C_{xz} - C_{yx}\right)^2}{C_x^2}}\left(\frac{C_{xz} - C_{yx}}{C_x^2}\right)\right]$$

Using $\omega_{5(Opt)}$ and $\omega_{6(Opt)}$, we get the following $MSE_{min}\left(\hat{\hat{Y}}_{MU,\gamma}\right)$;

$$MSE_{min}\left(\hat{\hat{Y}}_{MU,\gamma}\right) \cong \bar{Y}^2\left[1 - \frac{\psi C_{xz}^2}{4C_x^2} - \frac{\left\{1 + \left(\frac{3}{8} - \frac{\gamma}{4}\right)\psi C_z^2 - \frac{1}{2}\psi C_{yz} - \frac{\psi C_{xz}\left(C_{xz} - C_{yx}\right)}{2C_x^2}\right\}^2}{1 + \psi C_y^2 + \left(1 - \frac{\gamma}{2}\right)\psi C_z^2 - 2\psi C_{yz} - \frac{\psi\left(C_{xz} - C_{yx}\right)^2}{C_x^2}}\right] \qquad \ldots(10)$$

(6) Following the lines of Gupta and Shabbir (2008) and Grover and Kaur (2011), Shabbir and Gupta (2017) suggested the difference-cum-exponential ratio-type estimator as

$$\hat{\hat{Y}}_{SG} = [\omega_7\bar{y} + \omega_8(\bar{X} - \bar{x}) + \omega_9(\bar{Z} - \bar{z})]\exp\left(\frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}}\right) \qquad \ldots(11)$$

where ω_7, ω_8 and ω_9 are the feasible constants defined below;

$$\omega_{7(Opt)} = \frac{1 - \frac{1}{8}\psi C_x^2}{1 + \psi C_y^2\left(1 - R_{y.xz}^2\right)}$$

$$\omega_{8(Opt)} = R_1\left[\frac{\frac{1}{2}C_x(1 - \rho_{xz}^2)\left\{\psi C_y^2\left(1 - R_{y.xz}^2\right) - \left(1 - \frac{1}{4}\psi C_x^2\right)\right\} + C_y\left(\rho_{yx} - \rho_{yz}\rho_{xz}\right)\left(1 - \frac{1}{8}\psi C_x^2\right)}{C_x(1 - \rho_{xz}^2)\left\{1 + \psi C_y^2\left(1 - R_{y.xz}^2\right)\right\}}\right]$$

$$\omega_{9(Opt)} = R_4\left[\frac{C_y\left(\rho_{yz} - \rho_{yx}\rho_{xz}\right)\left(1 - \frac{1}{8}\psi C_x^2\right)}{C_x(1 - \rho_{xz}^2)\left\{1 + \psi C_y^2\left(1 - R_{y.xz}^2\right)\right\}}\right]$$

Substituting these optimal values, we have the minimum MSE given by,

$$MSE_{min}\left(\hat{\bar{Y}}_{SG}\right) \cong \bar{Y}^2\left[1-\frac{\left(1+\frac{1}{64}\psi^2C_x^4\right)+\frac{1}{4}\psi^2C_y^2C_x^2\left(1-R_{y.xz}^2\right)}{1+\psi C_y^2\left(1-R_{y.xz}^2\right)}\right] \quad ...(12)$$

Existing estimators in stratified random sampling

Consider $\Theta = \{\Theta_1, \Theta_2, \Theta_3, \ldots, \Theta_N\}$ be a finite population of size N and is divided into L homogenous strata with the h^{th} stratum containing $N_h, (h = 1,2,3,\ldots,L)$ units with the condition that $\sum_{h=1}^{L} N_h = N$. A sample size n_h is drawn under SRSWOR from the h^{th} stratum under the condition $\sum_{h=1}^{L} n_h = n$. Let $W_h = \left(\frac{N_h}{N}\right)$ be the

Table 1: Some useful measures in stratified random sampling

Measures	Study variable (y)	Auxiliary variable (x)	Auxiliary variable (z)
Sample mean	$\bar{y}_{st} = \sum_{h=1}^{L} W_h\bar{y}_h$	$\bar{x}_{st} = \sum_{h=1}^{L} W_h\bar{x}_h$	$\bar{z}_{st} = \sum_{h=1}^{L} W_h\bar{z}_h$
Sample mean (h^{th} stratum)	$\bar{y}_h = \sum_{i=1}^{n_h}\left(\frac{y_{hi}}{n_h}\right)$	$\bar{x}_h = \sum_{i=1}^{n_h}\left(\frac{x_{hi}}{n_h}\right)$	$\bar{z}_h = \sum_{i=1}^{n_h}\left(\frac{z_{hi}}{n_h}\right)$
Population mean	$\bar{Y} = \bar{Y}_{st} = \sum_{h=1}^{L} W_h\bar{Y}_h$	$\bar{X} = \bar{X}_{st} = \sum_{h=1}^{L} W_h\bar{X}_h$	$\bar{Z} = \bar{Z}_{st} = \sum_{h=1}^{L} W_h\bar{Z}_h$
Population mean (h^{th} stratum)	$\bar{Y}_h = \sum_{i=1}^{N_h}\left(\frac{y_{hi}}{N_h}\right)$	$\bar{X}_h = \sum_{i=1}^{N_h}\left(\frac{x_{hi}}{N_h}\right)$	$\bar{Z}_h = \sum_{i=1}^{N_h}\left(\frac{z_{hi}}{N_h}\right)$
Population standard deviation (h^{th} stratum)	$S_{yh} = \sqrt{(N_h-1)^{-1}\sum_{i=1}^{N_h}(y_{hi}-\bar{Y}_h)^2}$	$S_{xh} = \sqrt{(N_h-1)^{-1}\sum_{i=1}^{N_h}(x_{hi}-\bar{X}_h)^2}$	$S_{zh} = \sqrt{(N_h-1)^{-1}\sum_{i=1}^{N_h}(z_{hi}-\bar{Z}_h)^2}$
Population coefficient of variation (h^{th} stratum)	$C_{yh} = (\bar{Y}_h)^{-1}S_{yh}$	$C_{xh} = (\bar{X}_h)^{-1}S_{xh}$	$C_{zh} = (\bar{Z}_h)^{-1}S_{zh}$
Population correlation coefficients (h^{th} stratum)	$\rho_{yxh} = \left(S_{yh}S_{xh}\right)^{-1}S_{yxh}$ $\rho_{xzh} = \left(S_{xh}S_{zh}\right)^{-1}S_{xzh}$ $\rho_{yzh} = \left(S_{yh}S_{zh}\right)^{-1}S_{yzh}$		
Ratio of population means (h^{th} stratum)	$R_{1h} = \frac{\bar{Y}_h}{\bar{X}_h}$, $R_{2h} = \frac{\bar{X}_h}{\bar{Y}_h}$,	$R_{3h} = \frac{\bar{Z}_h}{\bar{Y}_h}$, $R_{4h} = \frac{\bar{Y}_h}{\bar{Z}_h}$	

known stratum weight and $\psi_h = \left(\frac{1}{n_h} - \frac{1}{N_h}\right)$ be the finite population correction factor. Other important measures are given in Table 1. Some well-known estimators for estimating the finite population mean using two auxiliary variables under stratified random sampling, from the literature are presented as follows;

(7) Usual mean estimator and its variance is given by,

$$\hat{\bar{Y}}_{O(st)} = \sum_{h=1}^{L} W_h \bar{y}_h \quad ...(13)$$

and

$$Var\left(\hat{\bar{Y}}_{O(st)}\right) = \sum_{h=1}^{L} W_h^2 \psi_h \bar{Y}_h^2 C_{yh}^2 = MSE\left(\hat{\bar{Y}}_{O(st)}\right) \quad ...(14)$$

(8) Traditional multivariate ratio estimator in stratified random sampling is given by,

$$\hat{\bar{Y}}_{MR(st)} = \sum_{h=1}^{L} W_h \bar{y}_h \left(\omega_{1h} \frac{\bar{X}_h}{\bar{x}_h} + \omega_{2h} \frac{\bar{Z}_h}{\bar{z}_h}\right) \quad ...(15)$$

where ω_{1h} and ω_{2h} are the suitable weights which satisfy the condition $\omega_{1h} + \omega_{2h} = 1$. The minimum MSE of $\hat{\bar{Y}}_{MR(st)}$ estimator to the first order of approximation is given as,

$$MSE_{min}\left(\hat{\bar{Y}}_{MR(st)}\right) \cong$$

$$\sum_{h=1}^{L} W_h^2 \psi_h \bar{Y}_h^2 \left(C_{yh}^2 + C_{zh}^2 - 2C_{yzh} - \frac{\left(C_{zh}^2 + C_{yxh} - C_{yzh} - C_{xzh}\right)^2}{C_{xh}^2 + C_{zh}^2 - 2C_{xzh}}\right) \quad ...(16)$$

The optimum values of ω_{1h} and ω_{2h} are given by,

$$\omega_{1h(opt)} = \frac{C_{zh}^2 + C_{yxh} - C_{yzh} - C_{xzh}}{C_{xh}^2 + C_{zh}^2 - 2C_{xzh}} \quad \text{and}$$

$$\omega_{2h(opt)} = 1 - \omega_{1h(opt)}$$

(9) The traditional multivariate regression estimator $\hat{\bar{Y}}_{MReg(st)}$ is given as,

$$\hat{\bar{Y}}_{MReg(st)} = \sum_{h=1}^{L} W_h \left[\bar{y}_h + b_{yxh}(\bar{X}_h - \bar{x}_h) + b_{yzh}(\bar{Z}_h - \bar{z}_h)\right] \quad ...(17)$$

The MSE of the estimator $\hat{\bar{Y}}_{MReg(st)}$ to the first order of approximation is given by,

$$MSE\left(\hat{\bar{Y}}_{MReg(st)}\right) \cong$$

$$\sum_{h=1}^{L} W_h^2 \psi_h \bar{Y}_h^2 C_{yh}^2 \left(1 - \rho_{yxh}^2 - \rho_{yzh}^2 + 2\rho_{yxh}\rho_{yzh}\rho_{xzh}\right) \quad ...(18)$$

(10) The traditional difference estimator is given by,

$$\hat{\bar{Y}}_{D(st)} = \sum_{h=1}^{L} W_h \left[\bar{y}_h + \omega_{3h}(\bar{X}_h - \bar{x}_h) + \omega_{4h}(\bar{Z}_h - \bar{z}_h)\right] \quad ...(19)$$

where ω_{3h} and ω_{4h} are the suitably chosen weights.

Minimum MSE of $\hat{\bar{Y}}_{D(st)}$ at optimum values of $\omega_{3h(opt)} = R_{1h}\left[\frac{C_{yh}(\rho_{yxh} - \rho_{yzh}\rho_{xzh})}{C_{xh}(1-\rho_{xzh}^2)}\right]$ and $\omega_{4h(opt)} = R_{4h}\left[\frac{C_{yh}(\rho_{yzh} - \rho_{yxh}\rho_{xzh})}{C_{zh}(1-\rho_{xzh}^2)}\right]$ is given below;

$$MSE_{min}\left(\hat{\bar{Y}}_{D(st)}\right) \cong \sum_{h=1}^{L} W_h^2 \psi_h \bar{Y}_h^2 C_{yh}^2 \left(1 - R_{y.xzh}^2\right) \quad ...(20)$$

where $R_{y.xzh}^2 = \dfrac{\rho_{yxh}^2 + \rho_{yzh}^2 - 2\rho_{yxh}\rho_{yzh}\rho_{xzh}}{1 - \rho_{xzh}^2}$

(11) On the lines of Gupta and Shabbir (2008) and Singh and Singh (2014), Muneer *et al*. (2017) proposed the following estimators;

$$\hat{\bar{Y}}_{MU,\gamma(st)} = \sum_{h=1}^{L} W_h \left[\{\omega_{5h}\bar{y}_h + \omega_{6h}(\bar{X}_h - \bar{x}_h)\}\right.$$

$$\left.\left[\gamma\left\{2 - \exp\left(\frac{\bar{z}_h - \bar{Z}_h}{\bar{z}_h + \bar{Z}_h}\right)\right\} + (1-\gamma)\exp\left(\frac{\bar{Z}_h - \bar{z}_h}{\bar{Z}_h + \bar{z}_h}\right)\right]\right] \quad ...(21)$$

where ω_{5h} and ω_{6h} are the constants to be determined. By using γ = 0 and 1, we obtained two estimators $\hat{\bar{Y}}_{MU,0}$ and $\hat{\bar{Y}}_{MU,1}$, respectively. The optimal values of ω_{5h} and ω_{6h} are stated below;

$$\omega_{5h(Opt)} = \frac{1+\left(\frac{3}{8}-\frac{\gamma}{4}\right)\psi_h C_{zh}^2 - \frac{1}{2}\psi_h C_{yzh} - \frac{\psi_h C_{xzh}\left(C_{xzh}-C_{yxh}\right)}{2C_{xh}^2}}{1+\psi_h C_{yh}^2 + \left(1-\frac{\gamma}{2}\right)\psi_h C_{zh}^2 - 2\psi_h C_{yzh} - \frac{\psi_h\left(C_{xzh}-C_{yxh}\right)^2}{C_{xh}^2}}$$

$$\omega_{6h(Opt)} = R_{1h}\left[\frac{C_{xzh}}{2C_{xh}^2} - \frac{1+\left(\frac{3}{8}-\frac{\gamma}{4}\right)\psi_h C_{zh}^2 - \frac{1}{2}\psi_h C_{yzh} - \frac{\psi_h C_{xzh}\left(C_{xzh}-C_{yxh}\right)}{2C_{xh}^2}}{1+\psi_h C_{yh}^2 + \left(1-\frac{\gamma}{2}\right)\psi_h C_{zh}^2 - 2\psi_h C_{yzh} - \frac{\psi_h\left(C_{xzh}-C_{yxh}\right)^2}{C_{xh}^2}}\left(\frac{C_{xzh}-C_{yxh}}{C_{xh}^2}\right)\right]$$

Using $\omega_{5h(Opt)}$ and $\omega_{6h(Opt)}$, we get the following $MSE_{min}\left(\hat{\bar{Y}}_{MU,\gamma(st)}\right)$ as,

$$MSE_{min}\left(\hat{\bar{Y}}_{MU,\gamma(st)}\right) \cong \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2\left[1-\frac{\psi_h C_{xzh}^2}{4C_{xh}^2} - \frac{\left\{1+\left(\frac{3}{8}-\frac{\gamma}{4}\right)\psi_h C_{zh}^2 - \frac{1}{2}\psi_h C_{yzh} - \frac{\psi_h C_{xzh}\left(C_{xzh}-C_{yxh}\right)}{2C_{xh}^2}\right\}^2}{1+\psi_h C_{yh}^2 + \left(1-\frac{\gamma}{2}\right)\psi_h C_{zh}^2 - 2\psi_h C_{yzh} - \frac{\psi_h\left(C_{xzh}-C_{yxh}\right)^2}{C_{xh}^2}}\right] \quad ...(22)$$

(12) Shabbir and Gupta (2017) followed the lines of Gupta and Shabbir (2008) and Grover and Kaur (2011) to suggest difference-cum-exponential ratio-type estimator as;

$$\hat{\bar{Y}}_{SG(st)} = \sum_{h=1}^{L} W_h\left[\left[\omega_{7h}\bar{y}_h + \omega_{8h}(\bar{X}_h - \bar{x}_h) + \omega_{9h}(\bar{Z}_h - \bar{z}_h)\right]\exp\left(\frac{\bar{X}_h - \bar{x}_h}{\bar{X}_h + \bar{x}_h}\right)\right] \quad ...(23)$$

where ω_{7h}, ω_{8h} and ω_{9h} are the feasible weights defined below;

$$\omega_{7h(Opt)} = \frac{1-\frac{1}{8}\psi_h C_{xh}^2}{1+\psi_h C_{yh}^2\left(1-R_{y.xzh}^2\right)}$$

$$\omega_{8h(Opt)} = R_{1h}\left[\frac{\frac{1}{2}C_{xh}\left(1-\rho_{xzh}^2\right)\left\{\psi_h C_{yh}^2\left(1-R_{y.xzh}^2\right) - \left(1-\frac{1}{4}\psi_h C_{xh}^2\right)\right\} + C_{yh}\left(\rho_{yxh}-\rho_{yzh}\rho_{xzh}\right)\left(1-\frac{1}{8}\psi_h C_{xh}^2\right)}{C_{xh}\left(1-\rho_{xzh}^2\right)\left\{1+\psi_h C_{yh}^2\left(1-R_{y.xzh}^2\right)\right\}}\right]$$

$$\omega_{9h(Opt)} = R_{4h}\left[\frac{C_{yh}\left(\rho_{yzh}-\rho_{yxh}\rho_{xzh}\right)\left(1-\frac{1}{8}\psi_h C_{xh}^2\right)}{C_{xh}\left(1-\rho_{xzh}^2\right)\left\{1+\psi_h C_{yh}^2\left(1-R_{y.xzh}^2\right)\right\}}\right]$$

At these optimal values, we have the minimum MSE given by,

$$MSE_{min}\left(\hat{\bar{Y}}_{SG(st)}\right) \cong \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2\left[1-\frac{\left(1+\frac{1}{64}\psi_h^2 C_{xh}^4\right)+\frac{1}{4}\psi_h^2 C_{yh}^2 C_{xh}^2\left(1-R_{y.xzh}^2\right)}{1+\psi_h C_{yh}^2\left(1-R_{y.xzh}^2\right)}\right] \quad ...(24)$$

METHODOLOGY

Haq and Shabbir (2014) introduced an exponential type estimator based on the information of single auxiliary variable as follows;

$$\hat{\bar{Y}}_{HS} = \left[\alpha_1\frac{1}{4}\bar{y}\left(\frac{\bar{X}}{\bar{x}}+\frac{\bar{x}}{\bar{X}}\right)\left\{\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right)+\exp\left(\frac{\bar{x}-\bar{X}}{\bar{X}+\bar{x}}\right)\right\} + \alpha_2(\bar{X}-\bar{x})\right]\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right)$$

Ekpenyong and Enang (2015) also suggested two exponential type ratio estimators using single auxiliary variable as given by,

$$\hat{\bar{Y}}_{EE1} = \alpha_3\bar{y} + \alpha_4(\bar{X} - \bar{x})\exp\left(\frac{(\bar{X} - \bar{x})}{\bar{X} + \bar{x}}\right)$$

$$\hat{\bar{Y}}_{EE2} = \alpha_5\bar{y} + \alpha_6(\bar{X} - \bar{x})\exp\left(\frac{2(\bar{X} - \bar{x})}{\bar{X} + \bar{x}}\right)$$

where $\alpha_i, i = 1,2,3,4,5,6$ are the suitably chosen weights.

Getting motivated from these recently developed efficient estimators, we combined some factors from $\hat{\bar{Y}}_{HS}$ and $\hat{\bar{Y}}_{EE1}$, and added the information of one more auxiliary variable to form our first proposed estimator. In the second proposed estimator, we added the information of one more auxiliary variable in $\hat{\bar{Y}}_{EE2}$ estimator. In this way, we proposed two new difference-cum-exponential ratio-type estimators to estimate the population mean $\bar{Y}$ based on two auxiliary variables under simple random sampling and extended this concept to stratified random sampling.

Proposed estimators in simple random sampling

(i) First proposed estimator:

$$\hat{\bar{Y}}_{P1} = \omega_{10}\left[\frac{1}{4}\bar{y}\left(\frac{\bar{X}}{\bar{x}} + \frac{\bar{x}}{\bar{X}}\right)\left\{\exp\left(\frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}}\right) + \exp\left(\frac{\bar{x} - \bar{X}}{\bar{X} + \bar{x}}\right)\right\}\right] + \omega_{11}(\bar{Z} - \bar{z}) + \omega_{12}(\bar{X} - \bar{x})\exp\left(\frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}}\right) \quad ...(25)$$

(ii) Second proposed estimator:

$$\hat{\bar{Y}}_{P2} = \omega_{13}\bar{y} + \omega_{14}(\bar{Z} - \bar{z}) + \omega_{15}(\bar{X} - \bar{x})\exp\left(\frac{2(\bar{X} - \bar{x})}{\bar{X} + \bar{x}}\right) \quad ...(26)$$

where $\omega_{10}, \omega_{11}, \ldots, \omega_{15}$ are the weights to be chosen, such that the MSE becomes minimum.

The following relative error terms along with their expectations are considered to obtain the expressions for bias, MSE and minimum MSE of the proposed estimators.

$$\left.\begin{aligned} \xi_o &= \tfrac{\bar{y}-\bar{Y}}{\bar{Y}} \\ \xi_1 &= \tfrac{\bar{z}-\bar{Z}}{\bar{Z}} \\ \xi_2 &= \tfrac{\bar{x}-\bar{X}}{\bar{X}} \end{aligned}\right\} \text{ such that } E(\xi_i) = 0, \text{ for } i = 0, 1 \text{ and } 2.$$

$E(\xi_0^2) = \psi C_y^2$, $\quad E(\xi_1^2) = \psi C_z^2$,
$E(\xi_0\xi_1) = \psi\rho_{yz}C_yC_z$, $\quad E(\xi_1\xi_2) = \psi\rho_{xz}C_xC_z$,

$E(\xi_2^2) = \psi C_x^2$
$E(\xi_0\xi_2) = \psi\rho_{yx}C_yC_x$

Biases of the proposed estimators $\hat{\bar{Y}}_{P1}$ and $\hat{\bar{Y}}_{P2}$, to the first order of approximation are given by,

$$Bias\left(\hat{\bar{Y}}_{P1}\right) \cong \bar{Y}\left[(\omega_{10} - 1) + \tau_1\left(\frac{\omega_{12}R_2}{2} + \frac{5\omega_{10}}{8}\right)\right] \quad ...(27)$$

and

$$Bias\left(\hat{\bar{Y}}_{P2}\right) \cong \bar{Y}[(\omega_{13} - 1) + \omega_{15}\tau_1R_2] \quad ...(28)$$

MSE of the proposed estimators are given below;

$$MSE\left(\hat{\bar{Y}}_{P1}\right) \cong \bar{Y}^2\left[1 + \omega_{10}^2\left(\tau_2 + \frac{5}{4}\tau_1\right) + \omega_{11}^2R_3^2\tau_3 + \omega_{12}^2R_2^2\tau_1 - \omega_{10}\left(2 + \frac{5}{4}\tau_1\right) - \omega_{12}R_2\tau_1 - 2\omega_{10}\omega_{11}R_3\tau_6 - \omega_{10}\omega_{12}R_2\tau_8 + 2\omega_{11}\omega_{12}R_2R_3\tau_7\right] \quad ...(29)$$

and

$$MSE\left(\hat{\bar{Y}}_{P2}\right) \cong \bar{Y}^2[1 + \omega_{13}^2\tau_2 + \omega_{14}^2R_3^2\tau_3 + \omega_{15}^2R_2^2\tau_1 - 2\omega_{13} - 2\omega_{15}R_2\tau_1 - 2\omega_{13}\omega_{14}R_3\tau_6 - 2\omega_{13}\omega_{15}R_2\tau_4 + 2\omega_{14}\omega_{15}R_2R_3\tau_7] \quad ...(30)$$

Partially differentiating $MSE\left(\hat{\bar{Y}}_{P1}\right)$ with respect to $\omega_i, i = 10, 11, 12$ and equating them to zero, we obtained the optimal values of ω_{10}, ω_{11} and ω_{12}. Similarly, differentiating $MSE\left(\hat{\bar{Y}}_{P2}\right)$ with respect to $\omega_i, i = 13, 14, 15$ and equating them to zero, we obtained the optimal values of ω_{13}, ω_{14} and ω_{15}. The optimal values of $\omega_i, i = 10,11,\ldots,15$ are given by,

$$\omega_{10(opt)} = \frac{A_1A_2 - 2\tau_1\tau_3A_3}{A_2A_4 - 2A_3^2}$$

$$\omega_{11(opt)} = \frac{2\tau_6(A_1A_2 - 2\tau_1\tau_3A_3) + \tau_7(A_1A_3 - \tau_1\tau_3A_4)}{2R_3\tau_3(A_2A_4 - 2A_3^2)}$$

$$\omega_{12(opt)} = \frac{\tau_1\tau_3A_4 - A_1A_3}{2R_2(A_2A_4 - 2A_3^2)}$$

$$\omega_{13(opt)} = \frac{\tau_2\tau_3A_5 - A_6(\tau_2\tau_7 + \tau_6A_7) + \tau_4A_7A_8}{\tau_2(A_5A_8 - A_6^2)}$$

$$\omega_{14(opt)} = \frac{\tau_6A_5 - A_6A_7}{R_3(A_5A_8 - A_6^2)}$$

$$\omega_{15(opt)} = \frac{A_7A_8 - \tau_6A_6}{R_2(A_5A_8 - A_6^2)}$$

Inserting optimal weights $[\omega_{10}, \omega_{11}$ and $\omega_{12}]$ in equation (29) and $[\omega_{13}, \omega_{14}$ and $\omega_{15}]$ in equation (30), we get the minimum MSE of the proposed estimators as follows;

$$MSE_{min}\left(\hat{\hat{Y}}_{P1}\right) \cong \frac{\overline{Y}^2}{4\tau_3B_1^2}[4\tau_3B_1^2 + (4\tau_2\tau_3 - 4\tau_6^2 + 5\tau_1\tau_3)B_2^2 + (\tau_1\tau_3 - \tau_7^2)B_3^2 - (8 + 5\tau_1)\tau_3B_1B_2 + 2(2\tau_6\tau_7 - \tau_3\tau_8)B_2B_3 - 2\tau_1\tau_3B_1B_3] \quad ...(31)$$

and

$$MSE_{min}\left(\hat{\hat{Y}}_{P2}\right) \cong \frac{\overline{Y}^2}{\tau_2B_4^2}[\tau_2B_4^2 + \tau_1\tau_2B_5^2 + \tau_2\tau_3B_6^2 + B_7^2 - 2(\tau_1B_4 - \tau_7B_6)\tau_2B_5 - 2(\tau_6B_6 + \tau_4B_5 + B_4)B_7] \quad ...(32)$$

where

$\tau_1 = \psi C_x^2,\qquad \tau_2 = 1 + \psi C_y^2,$

$\tau_3 = \psi C_z^2,\qquad \tau_4 = \psi C_x^2(\tau_5 - 1),$

$\tau_5 = \rho_{yx}\frac{C_y}{C_x},\qquad \tau_6 = \psi\rho_{yz}C_yC_z,$

$\tau_7 = \psi\rho_{xz}C_xC_z,\qquad \tau_8 = \psi C_x^2(2\tau_5 - 1),$

$A_1 = 8\tau_3 + 5\tau_1\tau_3,\qquad A_2 = \tau_1\tau_3 - \tau_7^2,$

$A_3 = 2\tau_6\tau_7 - \tau_3\tau_8,\qquad A_4 = 8\tau_2\tau_3 + 10\tau_1\tau_3 - 8\tau_6^2,$

$A_5 = \tau_1\tau_2 - \tau_4^2,\qquad A_6 = \tau_2\tau_7 - \tau_4\tau_6,$

$A_7 = \tau_1\tau_2 + \tau_4,\qquad A_8 = \tau_2\tau_3 - \tau_6^2,$

$B_1 = A_2A_4 - 2A_3^2,\qquad B_2 = A_1A_2 - 2\tau_1\tau_3A_3,$

$B_3 = \tau_1\tau_3A_4 - A_1A_3,\qquad B_4 = A_5A_8 - A_6^2,$

$B_5 = A_7A_8 - \tau_6A_6,\qquad B_6 = \tau_6A_5 - A_6A_7,$

$B_7 = \tau_2\tau_3A_5 - \tau_2\tau_7A_6 - \tau_6A_6A_7 + \tau_4A_7A_8.$

Proposed estimators under stratified random sampling

(i) First proposed estimator:

Following the lines of Haq and Shabbir (2014) and Ekpenyong and Enang (2015), we proposed a new exponential-type estimator as given by,

$$\hat{\hat{Y}}_{P1(st)} = \sum_{h=1}^{L} W_h\left[\omega_{10h}\left\{\frac{1}{4}\bar{y}_h\left(\frac{\bar{X}_h}{\bar{x}_h} + \frac{\bar{x}_h}{\bar{X}_h}\right)\{\exp(U_h) + \exp(-U_h)\}\right\} + \omega_{11h}(\bar{Z}_h - \bar{z}_h) + \omega_{12h}(\bar{X}_h - \bar{x}_h)\exp(U_h)\right] \quad ...(33)$$

where $U_h = \dfrac{\bar{X}_h - \bar{x}_h}{\bar{X}_h + \bar{x}_h}$

(ii) Second proposed estimator:

On the lines of Ekpenyong and Enang (2015), we proposed another exponential-type estimator given by,

$$\hat{\hat{Y}}_{P2(st)} = \sum_{h=1}^{L} W_h\left[\omega_{13h}\bar{y}_h + \omega_{14h}(\bar{Z}_h - \bar{z}_h) + \omega_{15h}(\bar{X}_h - \bar{x}_h)\exp\left(\frac{2(\bar{X}_h - \bar{x}_h)}{\bar{X}_h + \bar{x}_h}\right)\right] \quad ...(34)$$

where ω_{10h}, ω_{11h}, ω_{12h}, $\omega_{13h}, \omega_{14h}$ and ω_{15h} are the suitably chosen weights.

The following relative error terms and their expectations are used to derive the expressions for bias, MSE and minimum MSE of the proposed estimators;

$$\left.\begin{aligned} \xi_{oh} &= \frac{\bar{y}_h - \bar{Y}_h}{\bar{Y}_h} \\ \xi_{1h} &= \frac{\bar{z}_h - \bar{Z}_h}{\bar{Z}_h} \\ \xi_{2h} &= \frac{\bar{x}_h - \bar{X}_h}{\bar{X}_h} \end{aligned}\right\} \text{ such that } E(\xi_{ih}) = 0, \text{ for } i = 0,1 \text{ and } 2.$$

$E(\xi_{0h}^2) = \psi_h C_{yh}^2 = V_{200}$, $E(\xi_{1h}^2) = \psi_h C_{zh}^2 = V_{020}$,

$E(\xi_{2h}^2) = \psi_h C_{xh}^2 = V_{002}$,

$E(\xi_{0h}\xi_{1h}) = \psi_h \rho_{yzh} C_{yh} C_{zh} = V_{110}$,

$E(\xi_{1h}\xi_{2h}) = \psi_h \rho_{xzh} C_{xh} C_{zh} = V_{011}$,

$E(\xi_{0h}\xi_{2h}) = \psi_h \rho_{yxh} C_{yh} C_{xh} = V_{101}$

Bias of $\hat{\hat{Y}}_{P1(st)}$ and $\hat{\hat{Y}}_{P2(st)}$ to the first order of approximation are given below;

$$Bias\left(\hat{\hat{Y}}_{P1(st)}\right) \cong \sum_{h=1}^{L} W_h \bar{Y}_h \left[(\omega_{10h} - 1) + V_{002}\left(\frac{\omega_{12h} R_{2h}}{2} + \frac{5\omega_{10h}}{8}\right)\right] \quad ...(35)$$

and

$$Bias\left(\hat{\hat{Y}}_{P2(st)}\right) \cong \sum_{h=1}^{L} W_h \bar{Y}_h \left[(\omega_{13h} - 1) + \omega_{15h} V_{002} R_{2h}\right] \quad ...(36)$$

MSE of the proposed estimators $\hat{\hat{Y}}_{P1(st)}$ and $\hat{\hat{Y}}_{P2(st)}$, to the first order of approximation are given below:

$$MSE\left(\hat{\hat{Y}}_{P1(st)}\right) \cong \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 \Big[1 + \omega_{10h}^2\left(1 + V_{200} + \frac{5V_{002}}{4}\right) + \omega_{11h}^2 R_{3h}^2 V_{020} + \omega_{12h}^2 R_{2h}^2 V_{002} - \omega_{10h}\left(2 + \frac{5}{4}V_{002}\right) - \omega_{12h} R_{2h} V_{002} - 2\omega_{10h}\omega_{11h} R_{3h} V_{110} + 2\omega_{11h}\omega_{12h} R_{2h} R_{3h} V_{011} - \omega_{10h}\omega_{12h} R_{2h} V_{002}(2k - 1)\Big] \quad ...(37)$$

and

$$MSE\left(\hat{\hat{Y}}_{P2(st)}\right) \cong \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 \,[1 + \omega_{13h}^2(1 + V_{200}) + \omega_{14h}^2 R_{3h}^2 V_{020} + \omega_{15h}^2 R_{2h}^2 V_{002} - 2\omega_{13h} - 2\omega_{15h} R_{2h} V_{002} - 2\omega_{13h}\omega_{15h} R_{2h} V_{002}(k - 1) + 2\omega_{14h}\omega_{15h} R_{2h} R_{3h} V_{011} - 2\omega_{13h}\omega_{14h} R_{3h} V_{110}] \quad ...(38)$$

The optimal values $\omega_{ih}, i = 10,11,\ldots,15$ are given by,

$$\omega_{10h(opt)} = \frac{A_{1h}A_{2h} - 2V_{002}V_{020}A_{3h}}{A_{2h}A_{4h} - 2A_{3h}^2}$$

$$\omega_{11(opt)} = \frac{2V_{110}(A_{1h}A_{2h} - 2V_{002}V_{020}A_{3h}) + V_{011}(A_{1h}A_{3h} - V_{002}V_{020}A_{4h})}{2R_{3h}V_{020}(A_{2h}A_{4h} - 2A_{3h}^2)}$$

$$\omega_{12(opt)} = \frac{V_{002}V_{020}A_{4h} - A_{1h}A_{3h}}{2R_{2h}(A_{2h}A_{4h} - 2A_{3h}^2)}$$

$$\omega_{13h(opt)} = \frac{(1 + V_{200})V_{020}A_{5h} - A_{6h}(V_{011} + V_{011}V_{200} + V_{110}A_{7h}) + V_{002}(k - 1)A_{7h}A_{8h}}{(1 + V_{200})(A_{5h}A_{8h} - A_{6h}^2)}$$

$$\omega_{14h(opt)} = \frac{V_{110}A_{5h} - A_{6h}A_{7h}}{R_{3h}(A_{5h}A_{8h} - A_{6h}^2)}$$

$$\omega_{15h(opt)} = \frac{A_{7h}A_{8h} - V_{110}A_{6h}}{R_{2h}(A_{5h}A_{8h} - A_{6h}^2)}$$

Inserting optimal weights $\omega_{ih}, i = 10,11,\ldots,15$ in equations (37) and (38), we get the minimum MSE of the proposed estimators i.e. $\hat{\hat{Y}}_{P1(st)}$ and $\hat{\hat{Y}}_{P2(st)}$ as follows;

$$MSE_{min}\left(\hat{\hat{Y}}_{P1(st)}\right) \cong$$

$$\frac{\bar{Y}^2}{4V_{020}B_{1h}^2}\Big[4V_{020}B_{1h}^2 + (4V_{020} + 4V_{020}V_{200} - 4V_{110}^2 + 5V_{002}V_{020})B_{2h}^2 + (V_{002}V_{020} - V_{011}^2)B_{3h}^2 - (8 + 5V_{002})V_{020}B_{1h}B_{2h} + 2\big(2V_{110}V_{011} - V_{020}V_{002}(2k-1)\big)B_{2h}B_{3h} - 2V_{002}V_{020}B_{1h}B_{3h}\Big] \quad \ldots(39)$$

and

$$MSE_{min}\left(\hat{\hat{Y}}_{P2(st)}\right) \cong$$

$$\frac{\bar{Y}^2}{(1+V_{200})B_{4h}^2}\big[(1+V_{200})B_{4h}^2 + V_{002}(1+V_{200})B_{5h}^2 + V_{020}(1+V_{200})B_{6h}^2 + B_{7h}^2 - 2(V_{002}B_{4h} - V_{011}B_{6h})(1+V_{200})B_{5h} - 2(V_{110}B_{6h} + V_{002}(k-1)B_{5h} + B_{4h})B_{7h}\big] \quad \ldots(40)$$

where

$$k = \rho_{yxst}\frac{C_{yst}}{C_{xst}},$$

$$A_{1h} = 8V_{020} + 5V_{002}V_{020},$$

$$A_{2h} = V_{002}V_{020} - V_{011}^2,$$

$$A_{3h} = 2V_{110}V_{011} - V_{020}V_{002}(2k-1),$$

$$A_{4h} = 8V_{020}(1+V_{200}) + 10V_{002}V_{020} - 8V_{110}^2,$$

$$A_{5h} = V_{002}(1+V_{200}) - V_{002}^2(k-1)^2,$$

$$A_{6h} = V_{011}(1+V_{200}) - V_{110}V_{002}(k-1),$$

$$A_{7h} = V_{002}(1+V_{200}) + V_{002}(k-1),$$

$$A_{8h} = V_{020}(1+V_{200}) - V_{110}^2,$$

$$B_{1h} = A_{2h}A_{4h} - 2A_{3h}^2,$$

$$B_{2h} = A_{1h}A_{2h} - 2V_{002}V_{020}A_{3h},$$

$$B_{3h} = V_{002}V_{020}A_{4h} - A_{1h}A_{3h},$$

$$B_{4h} = A_{5h}A_{8h} - A_{6h}^2,$$

$$B_{5h} = A_{7h}A_{8h} - V_{110}A_{6h},$$

$$B_{6h} = V_{110}A_{5h} - A_{6h}A_{7h},$$

$$B_{7h} = V_{020}(1+V_{200})A_{5h} - V_{011}(1+V_{200})A_{6h} - V_{110}A_{6h}A_{7h} + V_{002}(k-1)A_{7h}A_{8h}.$$

RESULTS AND DISCUSSION

Numerical study in simple random sampling

We demonstrate the performances of the proposed and competing estimators through five natural datasets. Detailed description of each dataset is given below.

Dataset 1: Source: Koyuncu and Kadilar (2009)

The data for the illustration have been taken from 923 districts of 6 regions (Marmara, Agean, Mediterranean, Central Anatolia, Black Sea, East and Southeast Anatolia) in Turkey in 2007. Let y = number of teachers, x = number of students and z = number of classes both in primary and secondary schools.

The descriptive measures are:
$N = 923$, $n = 180$,
$\bar{Y} = 436.4345$, $\bar{X} = 11440.5$, $\bar{Z} = 333.1647$,
$C_y = 1.7183$, $C_x = 1.8645$, $C_z = 1.3280$,
$\rho_{yx} = 0.9543$, $\rho_{yz} = 0.9794$, $\rho_{xz} = 0.9465$

Dataset 2: Source: Ahmad (1997)

Let y = number of literate persons, x = number of cultivators and z = total population

The descriptive measures are:
$N = 376$, $n = 159$,
$\bar{Y} = 316.65$, $\bar{X} = 141.13$, $\bar{Z} = 1075.31$,

$C_y = 0.7721$, $C_x = 0.8450$, $C_z = 0.7746$,
$\rho_{yx} = 0.9106$, $\rho_{yz} = 0.9094$, $\rho_{xz} = 0.8614$

Dataset 3: Source: Abu-Dayyeh *et al*. (2003)

This data were taken from the District Handbook of Aligarh, India and is related to 332 villages. Let y = number of cultivators, x = area of village and z = number of households in a village.

The descriptive measures are:
$N = 332$, $n = 80$,
$\bar{Y} = 1093.10$, $\bar{X} = 181.57$, $\bar{Z} = 143.37$,
$C_y = 0.7626$, $C_x = 0.7684$, $C_z = 0.7616$,
$\rho_{yx} = 0.973$, $\rho_{yz} = 0.862$, $\rho_{xz} = 0.842$

Dataset 4: Source: Singh and Chaudhary (1986)

Let y = area under wheat in 1974, x = area under wheat in 1971 and z = area under wheat in 1973.

The descriptive measures are:
$N = 34$, $n = 20$,
$\bar{Y} = 856.41$, $\bar{X} = 208.88$, $\bar{Z} = 199.44$,
$C_y = 0.86$, $C_x = 0.72$, $C_z = 0.75$,
$\rho_{yx} = 0.45$, $\rho_{yz} = 0.45$, $\rho_{xz} = 0.98$

Dataset 5: Source: Cochran (1977)

Let y = number of placebo children, x = number of paralytic polio cases in the 'not in occulted' group and z = number of paralytic polio cases in the placebo group.

The descriptive measures are:
$N = 34$, $n = 15$,
$\bar{Y} = 4.92$, $\bar{X} = 2.59$, $\bar{Z} = 2.91$,
$C_y = 1.01232$, $C_x = 1.23187$, $C_z = 1.05351$,
$\rho_{yx} = 0.7326$, $\rho_{yz} = 0.643$, $\rho_{xz} = 0.6837$

The percentage relative efficiencies (PREs) of all the estimators with respect to usual unbiased estimator $\hat{\bar{Y}}_O$ are calculated through the expression given in equation (41).

$$PRE = \frac{MSE(\hat{\bar{Y}}_O)}{MSE(\hat{\bar{Y}}_i)} \times 100, \text{for } i = O, MR, MReg, D, MU_1,$$
$$MU_1, MU_0, SG, P1 \text{ and } P2 \qquad \ldots(41)$$

Table 2 provides the PREs of the proposed and competing estimators considered in the present study. Table 2 reveals that the proposed estimators, i.e. $\hat{\bar{Y}}_{P1}$ and $\hat{\bar{Y}}_{P2}$ have maximum PREs against all estimators in all datasets. These findings suggest that the proposed estimators are more efficient than the existing estimators.

Table 2: Percentage relative efficiencies (PREs) of estimators in simple random sampling

Estimators	Dataset 1	Dataset 2	Dataset 3	Dataset 4	Dataset 5
$\hat{\bar{Y}}_O$	100	100	100	100	100
$\hat{\bar{Y}}_{MR}$	1164.5160	872.4929	2123.6300	105.5530	195.0976
$\hat{\bar{Y}}_{MReg}$	103.8718	129.7944	138.3812	100.8166	144.0968
$\hat{\bar{Y}}_D$	1164.6500	907.1578	2127.8320	125.7143	235.0896
$\hat{\bar{Y}}_{MU,1}$	1159.6470	906.5428	1202.4890	126.6390	220.4138
$\hat{\bar{Y}}_{MU,0}$	1164.3120	907.5504	1207.8530	127.3708	225.6525
$\hat{\bar{Y}}_{SG}$	1174.4470	908.3624	2135.4120	127.5962	243.0908
$\hat{\bar{Y}}_{P2}$	**1384.4790**	**914.8019**	**2355.4150**	**128.9184**	**264.7184**
$\hat{\bar{Y}}_{P1}$	**1514.4600**	**930.3748**	**2484.3000**	**130.2963**	**307.2392**

Numerical study under stratified random sampling

We considered a real population of Turkey (2007) used by Koyuncu and Kadilar (2009). In this dataset y = number of teachers (study variable), x = number of students (first auxiliary variable) and z = number of classes (second auxiliary variable) recorded for primary and secondary schools at 6 regions for N = 923 districts. A total sample of size n = 180 is selected through Neyman allocation from 6 strata. The necessary data statistics are given in Table 3. PREs are calculated for this population to see the performance

Table 3: Population parameters

Values	h^{th} Stratum					
	1	2	3	4	5	6
N_h	127	117	103	170	205	201
n_h	31	21	29	38	22	39
$\bar{Y}_h$	703.7402	413.00	573.17	424.66	267.03	393.84
$\bar{X}_h$	20804.5906	9211.79	14309.30	9478.85	5569.95	12997.59
$\bar{Z}_h$	498.28	318.33	431.36	311.32	227.20	313.71
S_{yh}	883.8348	644.922	1033.467	810.585	403.654	711.723
S_{xh}	30486.7514	15180.769	27549.697	18218.931	8497.776	23094.141
S_{zh}	555.58	365.46	612.95	458.03	260.85	397.05
ρ_{yxh}	0.9366	0.9956	0.9938	0.9835	0.9893	0.9652
ρ_{yzh}	0.9790	0.9760	0.9840	0.9830	0.9640	0.9830
ρ_{xzh}	0.9396	0.9696	0.9770	0.9640	0.9676	0.9960

Table 4: Percentage relative efficiencies (PREs) of estimators in stratified random sampling

Estimators	$\hat{\bar{Y}}_{O(st)}$	$\hat{\bar{Y}}_{MR(st)}$	$\hat{\bar{Y}}_{MReg(st)}$	$\hat{\bar{Y}}_{D(st)}$	$\hat{\bar{Y}}_{MU,1(st)}$
PRE's	100.000	1923.821	110.208	4678.240	2058.423
Estimators	$\hat{\bar{Y}}_{MU,0(st)}$	$\hat{\bar{Y}}_{SG(st)}$	$\hat{\bar{Y}}_{P2(st)}$	$\hat{\bar{Y}}_{P1(st)}$	
PRE's	2063.828	5089.850	**5747.415**	**9971.986**	

of the proposed estimators, i.e. $\left(\hat{\bar{Y}}_{P1(st)} \text{ and } \hat{\bar{Y}}_{P2(st)}\right)$ as compared to other estimators under study, i.e. $\hat{\bar{Y}}_{O(st)}, \hat{\bar{Y}}_{MR(st)}, \hat{\bar{Y}}_{MReg(st)}, \hat{\bar{Y}}_{D(st)}, \hat{\bar{Y}}_{MU,1(st)}, \hat{\bar{Y}}_{MU,0(st)}$ and $\hat{\bar{Y}}_{SG(st)}$.

$$\text{where } PRE = \frac{MSE(\hat{\bar{Y}}_O)}{MSE(\hat{\bar{Y}}_{i(st)})} \times 100,$$

$$\text{for } i = O, MR, MReg, D, MU_1, MU_0, SG, P1 \text{ and } P2 \qquad \ldots(42)$$

Table 4 clearly shows that the proposed estimators are more efficient among all traditional and existing estimators as they have higher PRE values.

CONCLUSION

In this manuscript, ratio-type estimators for estimating population mean are proposed using linear combination of two auxiliary variables under simple and stratified random sampling. Numerical illustration through different datasets turned out that the proposed estimators are more competent than other estimators under study. Thus, survey statisticians may be encouraged for the practical application of the proposed estimators, if bivariate auxiliary information is available.

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