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RESEARCH ARTICLE

Estimation of the parameters in two linear models with some of the identical parameter vectors under the Pitman's closeness criterion

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Abstract: Two normal linear models for some of the identical parameters are discussed in this article. Many authors have studied the properties of estimators in two normal linear models for some of the identical parameters using mean squared error and mean squared error matrix criteria. In this article we give comparison between the estimators in two normal linear models with some identical parameters under Pitman's closeness criterion with known variances. It is noted that the estimators in two linear models with some identical parameters with known variances are superior over the estimators in single linear models with some identical parameters with known variances. Finally, the simulation is carried out to show that the theoretical results which we obtained in this study are aligned with different simulation conditions. The simulation results agree with our theoretical results.

Keywords: Best linear unbiased estimator, common parameters, mean squared error, Pitman's closeness criterion, two normal linear models.

INTRODUCTION

Consider the following system (H), which consists of two linear models:

$$y_1 = X_1\beta + Z_1\beta_1 + \varepsilon_1 \quad ...(1)$$

$$y_2 = X_2\beta + Z_2\beta_2 + \varepsilon_2 \quad ...(2)$$

where y_i shows a $n_i \times 1$ vector of observations, $(i = 1, 2)$, X_i and Z_i present $n_i \times p$ and $n_i \times t_i$ full rank matrices satisfying rank(X_i, Z_i)= rank(X_i) +rank (Z_i) for rank(.) stands for the rank of a matrix. β and β_i show $n \times 1$ and $t_i \times 1$ unknown parameters, ε_i present $n_i \times 1$ random vector supposed to satisfy a multivariate normal distribution with mean 0 and variance covariance matrix $\sigma_i I$, σ_i are known parameters, and ε_1 is independent of ε_2.

Define $Q_i = I_{ni} - Z_i(Z_i'Z_i)^{-1}Z_i'$, $T_i = (Z_i'Z_i)^{-1}Z_i'X_i$ and $r = \frac{\sigma_1}{\sigma_2}$, then by Liu (1996), we have the following results:

(1) For the single equation (1), the best linear unbiased estimator (BLUE) of β and β_1 are defined by,

$$\hat{\beta} = (X_1'Q_1X_1)^{-1}X_1'Q_1y_1 \quad ...(3)$$

$$\hat{\beta}_1 = (Z_1'Z_1)^{-1}Z_1'y_1 - T_1\hat{\beta} \quad ...(4)$$

(2) For the single equation (2), the BLUE of β and β_2 are presented by,

$$\tilde{\beta} = (X_2'Q_2X_2)^{-1}X_2'Q_2y_2 \quad ...(5)$$

$$\tilde{\beta}_2 = (Z_2'Z_2)^{-1}Z_2'y_2 - T_2\tilde{\beta} \quad ...(6)$$

(3) For the system (H), the BLUE of β, β_1 and β_2 are presented by,

$$\beta^*(r) = (X_1'Q_1X_1 + rX_2'Q_2X_2)^{-1}(X_1'Q_1y_1 + rX_2'Q_2y_2) \quad ...(7)$$

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$$\beta_1^* = (Z_1'Z_1)^{-1}Z_1'y_1 - T_1\beta^*(r) \quad ...(8)$$

$$\beta_2^* = (Z_2'Z_2)^{-1}Z_2'y_2 - T_2\beta^*(r) \quad ...(9)$$

In this article, we only study the estimation of the parameter β. Liu (1996) has presented the comparison between the estimators $\hat{\beta}$, $\tilde{\beta}$ and $\beta^*(r)$ in the mean squared error criterion when σ_i is known. He also gave an estimator when σ_i is unknown, and discussed the statistical properties of the estimators $\hat{\beta}$, $\tilde{\beta}$ and $\beta^*(r)$. Ma and Wang (2009) also discussed the estimators $\hat{\beta}$, $\tilde{\beta}$ and $\beta^*(r)$ in the mean squared error criterion.

Pitman closeness (PC) criterion was presented by Pitman (1937). Due to computation difficulty, it was used very rarely. After a meeting between Rao (1981) and Keating *et al*. (1993) to discuss the PC criterion, the PC criterion has obtained considerable attention as an important approach to compare the estimators. Since a meeting held by Rao and Keating. In literature, many authors have compared estimators using PC criterion. Wang and Yang (1994) used the PC criterion to compare two linear estimators in linear regression model and Reif (2006) used PC criterion to compare general pre-test estimators with some regression estimators. Yang *et al*. (2010) used it to compare two unified biased estimators in the linear regression model. Ahmadi and Balakrishnan (2009; 2010) used it to compare some order statistics, while Jozani (2014) studied the PC using the balanced loss function. Li *et al*. (2012) used the PC criterion to compare the *r – k* class estimator with the ordinary least squares estimator in linear regression model. Wu (2014) compared the modified *r – k* class estimator with the ordinary least squares estimator in linear regression model under PC criterion. Wu (2017) also compared the estimators under the PC criterion.

Liu (1996) gave the comparison between the estimators $\hat{\beta}$, $\tilde{\beta}$ and $\beta^*(r)$ in the mean squared error criterion when σ_i is known, however the comparison between the estimators $\hat{\beta}$, $\tilde{\beta}$ and $\beta^*(r)$ in the PC criterion was not done. In this study, we provide the comparison among the $\hat{\beta}$, $\tilde{\beta}$ and $\beta^*(r)$ in the PC criterion when σ_i is known.

RESULTS

First, we present some definitions which are needed to prove some of the obtained results.

Definition 2.1. Suppose that $\hat{\theta}_1$ and $\hat{\theta}_2$ are two estimators of the unknown p-dimensional vector θ. The PC of $\hat{\theta}_1$relative to $\hat{\theta}_2$ to estimate θ under a loss function $L(.,\theta)$ is denoted as $PC(\hat{\theta}_1,\hat{\theta}_2,\theta) = P_r(\hat{\theta}_1,\hat{\theta}_2,\theta) = P_r(\Delta(\hat{\theta}_1,\hat{\theta}_2) \geq 0)$, where

$$\Delta(\hat{\theta}_1,\hat{\theta}_2) = L(\hat{\theta}_2,\theta) - L(\hat{\theta}_1,\theta) \quad ...(10)$$

In this article, we study the quadratic loss function $L(\hat{\theta},\theta) = (\hat{\theta}-\theta)'U(\hat{\theta}-\theta)$ with a positive definite matrix, u.

Definition 2.2. $\hat{\theta}_1$ is said to be better than $\hat{\theta}_2$ for all $\theta \in \Theta$ in PC [under the loss function $L(.,\theta)$, with some parameter space Θ], if

$$PC(\hat{\theta}_1,\hat{\theta}_2,\theta) = P_r(\hat{\theta}_1,\hat{\theta}_2,\theta) = P_r(\Delta(\hat{\theta}_1,\hat{\theta}_2) \geq 0) \geq \frac{1}{2},\ for\ all\ \theta \in \Theta \quad ...(11)$$

Comparison of the estimator $\hat{\beta}$ and the estimator $\beta^*(r)$ under the PC criterion

Now we give the comparison of the estimator $\beta^*(r)$ given in equation (3) and the estimator $\beta^*(r)$ given in equation (7) under the PC criterion.

Theorem 2.1. Let σ_1, σ_2 are known and $\sigma_2 \leq 2\sigma_1$, then the estimator $\beta^*(r)$ given in equation (7) is superior over the estimator $\hat{\beta}$ given in equation (3) under the PC criterion.

Proof. Since matrices $X_1'Q_1X_1$ and $X_2'Q_2X_2$ are positive definite matrices (Liu, 1996), they can be diagonalised simultaneously, that is to say, there exists a reversible matrix H such that,

$$H'X_1'Q_1X_1H = I_p \quad ...(12)$$

and

$$H'X_2'Q_2X_2H = diag(m_1,...,m_p) = M_p \quad ...(13)$$

for $m_1,...,m_p$ present the positive eigenvalues of matrix $X_2'Q_2X_2(X_1'Q_1X_1)^{-1}$. Define:

$$H'X_1'Q_1\varepsilon_1 = \nu_1 \quad ...(14)$$

$$rM_p^{-1/2}H'X_2'Q_2\varepsilon_2 = \tau_1 \quad ...(15)$$

with ν_1 and τ_1 being p-dimension column random vector. It is easy to calculate that $\nu_1 \sim N(0,\sigma_1 I_p)$ and $\tau_1 \sim N(0,\sigma_1 I_p)$, and ν_1 is independent of τ_1. Choose $U = X_1'Q_1X_1$. Since $Q_iZ_i = 0,\ i = 1,2$, by definition 2.1, we have,

$$\begin{aligned} L(\hat{\beta},\beta) &= (\hat{\beta}-\beta)'X_1'Q_1X_1(\hat{\beta}-\beta) \\ &= ((X_1'Q_1X_1)^{-1}X_1'Q_1y_1-\beta)'X_1'Q_1X_1((X_1'Q_1X_1)^{-1}X_1'Q_1y_1-\beta) \\ &= (H'X_1'Q_1\varepsilon_1)'(H'X_1'Q_1\varepsilon_1) = \nu_1'\nu_1 \end{aligned} \quad ...(16)$$

$$\begin{aligned} L(\beta^*(r),\beta) &= (\beta^*(r)-\beta)'X_1'Q_1X_1(\beta^*(r)-\beta) \\ &= ((X_1'Q_1X_1+rX_2'Q_2X_2)^{-1}(X_1'Q_1y_1+rX_2'Q_2y_2)-\beta)'X_1'Q_1X_1 \\ &\quad \times((X_1'Q_1X_1+rX_2'Q_2X_2)^{-1}(X_1'Q_1y_1+rX_2'Q_2y_2)-\beta) \\ &= (H'X_1'Q_1\varepsilon_1+rH'X_2'Q_2\varepsilon_2)'(I_p+rM_p)^{-2} \\ &\quad \times(H'X_1'Q_1\varepsilon_1+rH'X_2'Q_2\varepsilon_2) \\ &= (\nu_1+M_p^{1/2}\tau_1)'(I_p+rM_p)^{-2}(\nu_1+M_p^{1/2}\tau_1) \\ &= \nu_1'(I_p+rM_p)^{-2}\nu_1+\nu_1'M_p^{1/2}(I_p+rM_p)^{-2}\tau_1 \\ &\quad +\tau_1'M_p^{1/2}(I_p+rM_p)^{-2}\nu_1+\tau_1'M_p(I_p+rM_p)^{-2}\tau_1 \end{aligned} \quad ...(17)$$

Then we have,

$$\begin{aligned} PC(\beta^*(r),\hat{\beta},\beta) &= P_r(L(\beta^*(r),\beta)\leq L(\hat{\beta},\beta)) \\ &= P_r\{\nu_1'(I_p+rM_p)^{-2}\nu_1+\nu_1'M_p^{1/2}(I_p+rM_p)^{-2}\tau_1 \\ &\quad +\tau_1'M_p^{1/2}(I_p+rM_p)^{-2}\nu_1+\tau_1'M_p(I_p+rM_p)^{-2}\tau_1\leq\nu_1'\nu_1\} \\ &= P_r\{\nu_1'((I_p+rM_p)^{-2}-I)\nu_1+\nu_1'M_p^{1/2}(I_p+rM_p)^{-2}\tau_1 \\ &\quad +\tau_1'M_p^{1/2}(I_p+rM_p)^{-2}\nu_1+\tau_1'M_p(I_p+rM_p)^{-2}\tau_1\leq 0\} \\ &= P_r\{-\nu_1'(I_p+rM_p)^{-2}(2rM_p+r^2M_p^2)\nu_1+\nu_1'M_p^{1/2}(I_p+rM_p)^{-2}\tau_1 \\ &\quad +\tau_1'M_p^{1/2}(I_p+rM_p)^{-2}\nu_1+\tau_1'M_p(I_p+rM_p)^{-2}\tau_1\leq 0\} \\ &\geq P_r\{-2r\nu_1'(I_p+rM_p)^{-2}M_p\nu_1+\nu_1'M_p^{1/2}(I_p+rM_p)^{-2}\tau_1 \\ &\quad +\tau_1'M_p^{1/2}(I_p+rM_p)^{-2}\nu_1+\tau_1'M_p(I_p+rM_p)^{-2}\tau_1\leq 0\} \end{aligned} \quad ...(18)$$

Since $r \geq \frac{1}{2}$, we have $2r \geq 1$, that is $\sigma_2 \leq 2\sigma_1$. Then we obtain the following:

$$\begin{aligned} &PC(\beta^*(r),\hat{\beta},\beta)\geq \\ &P_r\{-\nu_1'(I_p+rM_p)^{-2}M_p\nu_1+\nu_1'M_p^{1/2}(I_p+rM_p)^{-2}\tau_1 \\ &+\tau_1'M_p^{1/2}(I_p+rM_p)^{-2}\nu_1+\tau_1'M_p(I_p+rM_p)^{-2}\tau_1\leq 0\} \end{aligned} \quad ...(19)$$

Now we compute,

$$\begin{aligned} &PC(\beta^*(1),\hat{\beta},\beta)\geq \\ &P_r\{-\nu_1'(I_p+rM_p)^{-2}M_p\nu_1+\nu_1'M_p^{1/2}(I_p+rM_p)^{-2}\tau_1 \\ &+\tau_1'M_p^{1/2}(I_p+rM_p)^{-2}\nu_1+\tau_1'M_p(I_p+rM_p)^{-2}\tau_1\leq 0\}. \end{aligned} \quad ...(20)$$

Denote $\eta_1 = \frac{1}{\sigma_1^{-1/2}}\nu_1$, $\eta_2 = \frac{1}{\sigma_1^{-1/2}}\tau_1$, then $\eta_1 \sim N(0,I)$ and $\eta_2 \sim N(0,I)$. Therefore, equation (20) can be reduced to:

$$\begin{aligned} &PC(\beta^*(1),\hat{\beta},\beta)\geq \\ &P_r\{-\eta_1'(I_p+rM_p)^{-2}M_p\eta_1+\eta_1'M_p^{1/2}(I_p+rM_p)^{-2}\eta_2 \\ &+\eta_2'M_p^{1/2}(I_p+rM_p)^{-2}\eta_1+\eta_2'M_p(I_p+rM_p)^{-2}\eta_2\leq 0\} \end{aligned} \quad ...(21)$$

Since the normal distribution is a spherically symmetric distribution (Ma & Wang, 2009), by the properties of spherically symmetric distribution, $\psi_1 = \eta_1'(I_p+r_1M_p)^{-2}M_p\eta_1$ and $R_1 = \frac{1}{(\eta_1'(I_p+rM_p)^{-2}M_p\eta_1)^{\frac{1}{2}}}\eta_1$ are independent. Similarly, $\psi_2 = \eta_2'(I_p+rM_p)^{-2}M_p\eta_2$ and $R_2 = \frac{1}{(\eta_2'(I_p+rM_p)^{-2}M_p\eta_2)^{\frac{1}{2}}}\eta_2$ are independent. ψ_1, ψ_2 have the same distribution and R_1, R_2 have the same distribution.

Then, equation (21) can be reduced to:

$$P_r\{-\psi_1+(\psi_1\psi_2)^{\frac{1}{2}}R_1'(I_p+rM_p)^{-2}M_p^{\frac{1}{2}}R_2$$
$$+(\psi_1\psi_2)^{\frac{1}{2}}R_2'(I_p+rM_p)^{-2}M_p^{\frac{1}{2}}R_1+\psi_2\leq 0\} \quad ...(22)$$

Since R_1 and $-R_1$ have the same distribution, ψ_1 is independent from ψ_2 and they also have the same distribution, we can change R_1 as $-R_1$ and change ψ_1 as ψ_2 in equation (22), then we obtain,

$$\begin{aligned} & P_r\{-\psi_1+(\psi_1\psi_2)^{\frac{1}{2}}R_1'(I_p+rM_p)^{-2}M_p^{\frac{1}{2}}R_2 \\ & +(\psi_1\psi_2)^{\frac{1}{2}}R_2'(I_p+rM_p)^{-2}M_p^{\frac{1}{2}}R_1+\psi_2\leq 0\} \\ = & P_r\{-\psi_2-(\psi_1\psi_2)^{\frac{1}{2}}R_1'(I_p+rM_p)^{-2}M_p^{\frac{1}{2}}R_2 \\ & -(\psi_1\psi_2)^{\frac{1}{2}}R_2'(I_p+rM_p)^{-2}M_p^{\frac{1}{2}}R_1+psi_1\leq 0\} \\ = & 1-P_r\{-\psi_2-(\psi_1\psi_2)^{\frac{1}{2}}R_1'(I_p+rM_p)^{-2}M_p^{\frac{1}{2}}R_2 \\ & -(\psi_1\psi_2)^{\frac{1}{2}}R_2'(I_p+rM_p)^{-2}M_p^{\frac{1}{2}}R_1+\psi_1\geq 0\} \\ = & 1-P_r\{-\psi_1+(\psi_1\psi_2)^{\frac{1}{2}}R_1'(I_p+rM_p)^{-2}M_p^{\frac{1}{2}}R_2 \\ & +(\psi_1\psi_2)^{\frac{1}{2}}R_2'(I_p+rM_p)^{-2}M_p^{\frac{1}{2}}R_1+\psi_2\leq 0\} \quad ...(23)\end{aligned}$$

By equation (23), we obtain;

$$P_r\{-\psi_1+(\psi_1\psi_2)^{\frac{1}{2}}R_1'(I_p+rM_p)^{-2}M_p^{\frac{1}{2}}R_2$$
$$+(\psi_1\psi_2)^{\frac{1}{2}}R_2'(I_p+rM_p)^{-2}M_p^{\frac{1}{2}}R_1+\psi_2\leq 0\}=\frac{1}{2} \quad ...(24)$$

Thus, we have,

$$PC(\beta^*(r),\hat{\beta},\beta)\geq\frac{1}{2} \quad ...(25)$$

That is, when $\sigma_i,\ i=1,2$ are known and $\sigma_2\leq 2\sigma_1$; the estimator $\beta^*(r)$ is superior over the estimator $\hat{\beta}$ under the PC criterion.

Comparison of the estimator $\hat{\beta}$ and the estimator $\beta^*(r)$ under the PC criterion

Now we compare the estimator $\tilde{\beta}$ given in equation (5) and the estimator $\beta^*(r)$ given in equation (7) under the PC criterion.

Theorem 2.2. When σ_1,σ_2 are known and $\sigma_1\leq 2\sigma_2$, then the estimator $\beta^*(r)$ given in equation (7) is superior over the estimator $\tilde{\beta}$ given in equation (5) in the PC criterion.

Proof. Matrices $X_1'Q_1X_1$ and $X_2'Q_2X_2$ are positive definite and they can be diagonalised simultaneously, that is, there exists a reversible matrix G such that,

$$G'X_2'Q_2X_2G=I_p \quad ...(26)$$

and

$$G'X_1'Q_1X_1G=diag(s_1,...,s_p)=S_p \quad ...(27)$$

with $s_1,...,s_p$ showing the positive eigenvalues of matrix $X_1'Q_1X_1(X_2'Q_2X_2)^{-1}$. Define:

$$G'X_2'Q_2\varepsilon_2=\zeta_1 \quad ...(28)$$

$$\frac{1}{r}S_p^{-1/2}G'X_1'Q_1\varepsilon_1=\rho_1 \quad ...(29)$$

where ζ_1 and ρ_1 present p-dimension column random vectors. It is easy to compute that $\zeta_1\sim N(0,\sigma_2I_p)$ and $\rho_1\sim N(0,\sigma_2I_p)$, and ζ_1 is independent of ρ_1. Choose $U=X_2'Q_2X_2$. As $Q_iZ_i=0,\ i=1,2$, by definition 2.1, we have,

$$\begin{aligned} L(\tilde{\beta},\beta) &= (\tilde{\beta}-\beta)'X_2'Q_2X_2(\tilde{\beta}-\beta) \\ &= ((X_2'Q_2X_2)^{-1}X_2'Q_2y_2-\beta)'X_2'Q_2X_2((X_2'Q_2X_2)^{-1}X_2'Q_2y_2-\beta) \\ &= (G'X_2'Q_2\varepsilon_2)'(G'X_2'Q_2\varepsilon_2)=\zeta_1'\zeta_1 \quad ...(30)\end{aligned}$$

$$\begin{aligned} L(\beta^*(r),\beta) &= (\beta^*(r)-\beta)'X_2'Q_2X_2(\beta^*(r)-\beta) \\ &= ((X_1'Q_1X_1+rX_2'Q_2X_2)^{-1}(X_1'Q_1y_1+rX_2'Q_2y_2)-\beta)'X_2'Q_2X_2\times((X_1'Q_1X_1+rX_2'Q_2X_2)^{-1} \\ &\quad (X_1'Q_1y_1+rX_2'Q_2y_2)-\beta) \\ &= ((\frac{1}{r}X_1'Q_1X_1+X_2'Q_2X_2)^{-1}(X_1'Q_1y_1+\frac{1}{r}X_2'Q_2y_2)-\beta)'X_2'Q_2X_2\times((\frac{1}{r}X_1'Q_1X_1+X_2'Q_2X_2)^{-1} \\ &\quad (X_1'Q_1y_1+\frac{1}{r}X_2'Q_2y_2)-\beta) \\ &= (G'X_2'Q_2\varepsilon_2+\frac{1}{r}G'X_1'Q_1\varepsilon_1)'(I_p+\frac{1}{r}M_p)^{-2}\times(G'X_2'Q_2\varepsilon_2+\frac{1}{r}G'X_1'Q_1\varepsilon_1)\end{aligned}$$

$$= (\zeta_1 + S_p^{1/2}\rho_1)'(I_p + \frac{1}{r}S_p)^{-2}(\zeta_1 + S_p^{1/2}\rho_1)$$

$$= \zeta_1'(I_p + \frac{1}{r}S_p)^{-2}\zeta_1 + \zeta_1' S_p^{1/2}(I_p + \frac{1}{r}S_p)^{-2}\rho_1 + \rho_1' S_p^{1/2}(I_p + \frac{1}{r}S_p)^{-2}\zeta_1 + \rho_1' S_p(I_p + rS_p)^{-2}\rho_1 \quad ...(31)$$

Thus, we have,

$$\begin{aligned}
PC(\beta^*(r), \tilde{\beta}, \beta) &= P_r(L(\beta^*(r), \beta) \leq L(\tilde{\beta}, \beta)) \\
&= P_r\{\zeta_1'(I_p + \frac{1}{r}S_p)^{-2}\zeta_1 + \zeta_1' S_p^{1/2}(I_p + \frac{1}{r}S_p)^{-2}\rho_1 + \rho_1' S_p^{1/2}(I_p + \frac{1}{r}S_p)^{-2}\zeta_1 + \\
&\quad \rho_1' S_p(I_p + rS_p)^{-2}\rho_1 \leq \zeta_1'\zeta_1\} \\
&= P_r\{\zeta_1'[(I_p + \frac{1}{r}S_p)^{-2} - I]\zeta_1 + \zeta_1' S_p^{1/2}(I_p + \frac{1}{r}S_p)^{-2}\rho_1 + \rho_1' S_p^{1/2}(I_p + \frac{1}{r}S_p)^{-2}\zeta_1 + \\
&\quad \rho_1' S_p(I_p + rS_p)^{-2}\rho_1 \leq 0\} \\
&= P_r\{\zeta_1'(I_p + \frac{1}{r}S_p)^{-2}(-\frac{2}{r}S_p + \frac{1}{r^2}S_p^2)\zeta_1 + \zeta_1' S_p^{1/2}(I_p + \frac{1}{r}S_p)^{-2}\rho_1 + \rho_1' S_p^{1/2}(I_p + \frac{1}{r}S_p)^{-2}\zeta_1 + \\
&\quad \rho_1' S_p(I_p + rS_p)^{-2}\rho_1 \leq 0\} \\
&\geq P_r\{-\frac{2}{r}\zeta_1' S_p(I_p + \frac{1}{r}S_p)^{-2}\zeta_1 + \zeta_1' S_p^{1/2}(I_p + \frac{1}{r}S_p)^{-2}\rho_1 + \rho_1' S_p^{1/2}(I_p + \frac{1}{r}S_p)^{-2}\zeta_1 + \\
&\quad \rho_1' S_p(I_p + rS_p)^{-2}\rho_1 \leq 0\}
\end{aligned} \quad ...(32)$$

Since $\frac{2}{r} \geq 1$, we $r \leq 2$, that is $\sigma_1 \leq 2\sigma_2$, and we obtain,

$$\begin{aligned}
PC(\beta^*(r), \tilde{\beta}, \beta) \geq\ & P_r\{-\zeta_1' S_p(I_p + \frac{1}{r}S_p)^{-2}\zeta_1 + \zeta_1' S_p^{1/2}(I_p + \frac{1}{r}S_p)^{-2}\rho_1 \\
& + \rho_1' S_p^{1/2}(I_p + \frac{1}{r}S_p)^{-2}\zeta_1 + \rho_1' S_p(I_p + rS_p)^{-2}\rho_1 \leq 0\}
\end{aligned} \quad ...(33)$$

Then following the proof of Theorem 2.1, we get,

$$PC(\beta^*(r), \tilde{\beta}, \beta) \geq \frac{1}{2} \quad ...(34)$$

$\sigma_1,\ \sigma_2$ are known and $\sigma_1 \leq 2\sigma_2$, then the estimator $\beta^*(r)$ given in equation (7) is superior over the estimator $\tilde{\beta}$ given in equation (5) under the PC criterion.

A SIMULATION STUDY

In this section we present simulation results to validate the theoretical results. The models are given as follows:

$$y_1 = X_1\beta + Z_1\beta_1 + \varepsilon_1, \varepsilon_1 \sim N(0, \sigma_1 I_3) \quad ...(35)$$

$$y_2 = X_2\beta + Z_2\beta_2 + \varepsilon_2, \varepsilon_2 \sim N(0, \sigma_2 I_3) \quad ...(36)$$

where

$$X_1 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}, Z_1 = \begin{pmatrix} 5 & 6 \\ 7 & 8 \\ 9 & 10 \end{pmatrix}, \beta = \begin{pmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \end{pmatrix},$$

$$\beta_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad ...(37)$$

$$X_2 = \begin{pmatrix} 1 & 3 \\ 5 & 7 \\ 9 & 11 \end{pmatrix}, Z_2 = \begin{pmatrix} 10 & 18 \\ 25 & 30 \\ 40 & 60 \end{pmatrix}, \beta = \begin{pmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \end{pmatrix},$$

$$\beta_2 = \begin{pmatrix} 10 \\ 20 \\ 30 \end{pmatrix} \quad ...(38)$$

For the values of σ_1 and σ_2, see Tables 1and 2. Now we compute $PC(\beta^*(r), \hat{\beta}, \beta)$ and $PC(\beta^*(r), \tilde{\beta}, \beta)$. The simulation is replicated 10000 times.

By Table 1, the estimator $\beta^*(r)$ is superior to the estimator $\hat{\beta}$ under the PC criterion. By Table 2, the estimator $\beta^*(r)$ is superior to the estimator $\tilde{\beta}$ under the PC criterion. By Tables 1 and 2, we find that our simulation results agree with the Theorems 2.1 and 2.2.

Table 1: The estimated $PC(\beta^*(r), \hat{\beta}, \beta)$ for σ_1 = 0:1 and different values σ_2

σ_2	0.01	0.02	0.05	0.1	0.2
$PC(\beta^*(r), \hat{\beta}, \beta)$	0.9684	0.9668	0.9686	0.9678	0.9648

Table 2: The estimated $PC(\beta^*(r), \tilde{\beta}, \beta)$ for σ_2 = 0:1 and different values σ_1

σ_1	0.01	0.02	0.05	0.1	0.2
$PC(\beta^*(r), \tilde{\beta}, \beta)$	0.9328	0.9417	0.9536	0.9618	0.9672

CONCLUSION

In this article, we have compared the estimators in system H under the PC criterion. We have proven that the estimators in two linear equations in system H are superior over the estimators in a single equation system with known variances and under certain conditions. In future research, we will study the comparison among the estimators in system H under PC criterion when the variances are unknown.

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