

A RESULT ON TWO-DIMENSIONAL POLAR LATTICES

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Abstract : Suppose P and P^0 denote closed positive and open positive quadrants in R^2 respectively. Let Λ be any lattice in R^2 with polar lattice Λ^* . Let F be a convex and symmetric (with respect to the axes of coordinates) distance function with $F(1,0) = F(0,1) = 1$, where $t \in R$ and let $\mu = \text{area} \{ \underline{x} \in R^2 / F(\underline{x}) \leq 1 \}$. For certain distance functions F , there exist non-zero $\underline{x} \in P \cap \Lambda$ and $\underline{y} \in P^0 \cap \Lambda^*$ such that $\mu F(\underline{x}) F(\underline{y}) \leq \gamma_t$, where γ_t is a constant depending on t and the distance function. There exist a lower bound $2(t + 1/t)$ and an upper bound $4(t + 1/t)$ for γ_t over all convex symmetric distance functions.

1. Introduction

Let P and P^0 denote closed positive and open positive quadrants in R^2 respectively. Let Λ be any lattice in R^2 with polar lattice Λ^* . Let F be a convex and symmetric (with respect to the axes of coordinates) distance function with $F(1,0) = F(0,1) = 1$, where $t \in R$. Without loss of generality we can take $t \geq 1$. If $t \leq 1$, we have the same situation as in the case when $t \geq 1$ with the coordinate axes interchanged. Let $\mu = \text{area} \{ \underline{x} \in R^2 / f(\underline{x}) \leq 1 \}$.

Hossain and Worley³ have shown that for certain distance functions F , there exist non-zero $\underline{x} \in P \cap \Lambda$ and $\underline{y} \in P^0 \cap \Lambda^*$ such that

$$\mu F(\underline{x}) F(\underline{y}) \leq \gamma_t,$$

where γ_t is a constant depending on t and the distance function. In this note we show that γ_t has a lower bound and an upper bound over all the convex symmetric distance functions. In this note symmetric means the symmetry with respect to the axes of coordinates.

2. Discussion

The following notations will be used frequently in this section.

$$F_1(\Lambda) = \inf \{ F(\underline{x}) : \underline{x} \in \Lambda \cap P \}$$

$$F_2(\Lambda^*) = \inf \{ F(\underline{x}^*) : \underline{x}^* \in \Lambda^* \cap P^0 \},$$

where F is a distance function.

Theorem:

If F is any convex symmetric distance function with $F(1,0)=F(0,t)=1$, then

$$2(t+1/t) \leq \mu F_1(\Lambda) F_2(\Lambda^*) \leq 4(t+1/t)$$

for the lattice Λ with basis $\{(1,0), (0,t)\}$

The lower bound is best possible for the distance function $F(x_1, x_2) = |x_1| + 1/t |x_2|$ and the lattice Λ with a basis $(1,0)$ and $(0,t)$. The upper bound may not be best possible, but cannot be below $4t$. In order to prove the theorem, we use the following lemmas.²

Lemma 1

Let Λ be the lattice with a basis $\{(1,0), (0,t)\}$. Then $\min_{\alpha} \mu F_1(\Lambda) F_2(\Lambda^*) = 2(t+1/t)$, for the convex symmetric polygonal distance function F given by

$$F(x_1, x_2) = \max \left\{ \frac{1-\alpha}{\alpha} |x_1| + 1/t |x_2|, |x_1| + \frac{1-\alpha}{\alpha t} |x_2| \right\}$$

where $1/2 \leq \alpha \leq 1$.

(α has to satisfy the above conditions since F is convex and symmetric).

Lemma 2

Let Λ be a lattice with basis $\{(1,0), (0,t)\}$. Let F be the convex polygonal distance function, where $F(x_1, x_2) = 1$ has two more vertices at $(\alpha, \alpha t)$ and $(\beta, \beta/t)$ in P^0 in addition to $(1,0)$ and $(0,t)$, where $1/2 \leq \alpha \leq 1$ and the limit of β depends on α .

$$\text{Then } \min_{\alpha, \beta} \mu F_1(\Lambda) F_2(\Lambda^*) \geq 2(t+1/t).$$

From Lemma 1 and Lemma 2, we can establish the left hand side of the inequality in the theorem.

Suppose $F(x_1, x_2) = 1$ intersects the

lines OL at B and OM at C respectively,

where $L \equiv (1,t)$ and $M \equiv (1,1/t)$.

Let $B \equiv (\alpha, \alpha t)$ and $C \equiv (\beta, \beta/t)$.

The curve $F(x_1, x_2) = 1$ passes through

the points $A \equiv (0,t)$ and $D \equiv (1,0)$.

Then $F_1(\Lambda) = 1$.

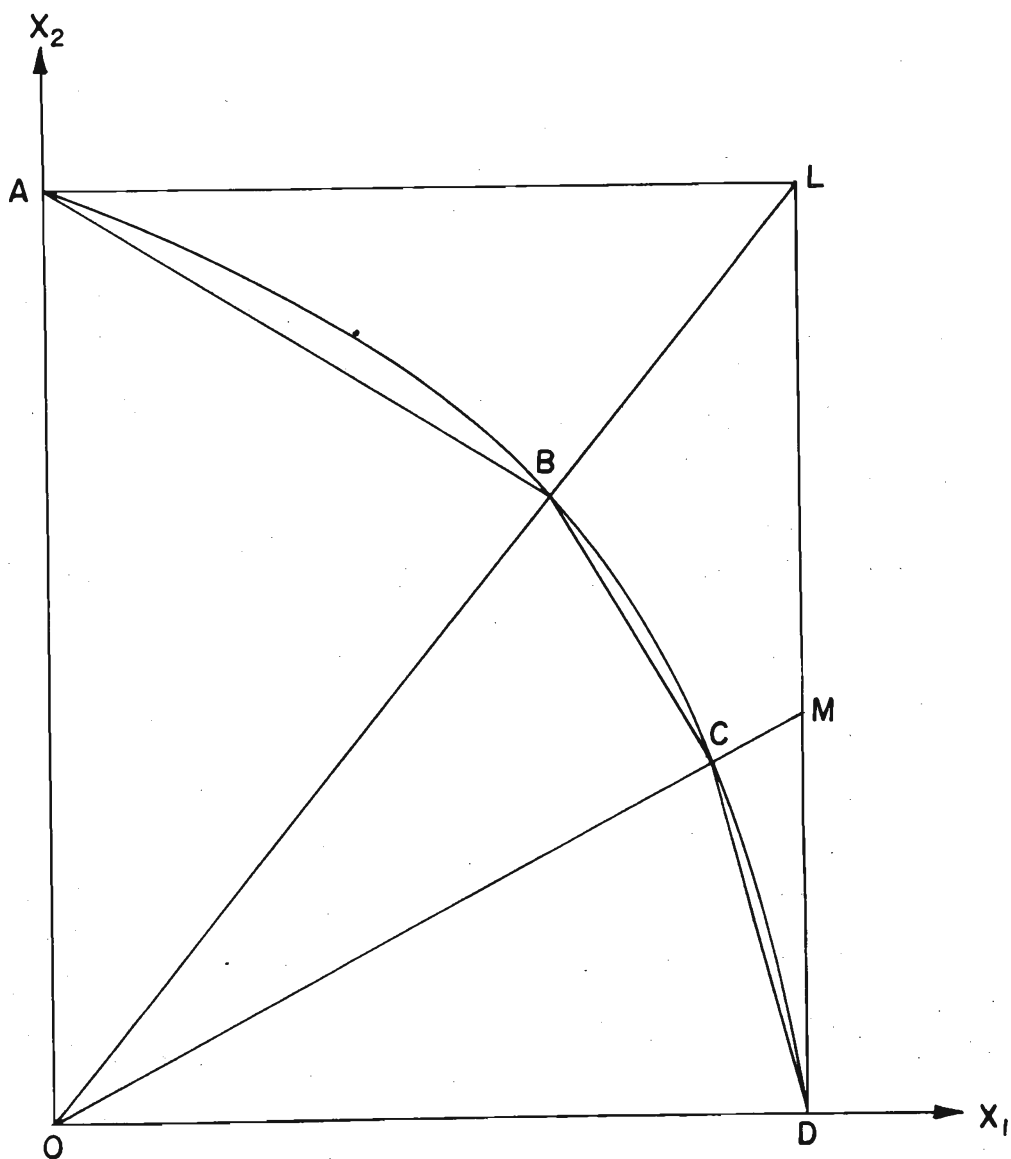


Figure 1. The curve $F(x_1, x_2) = 1$ passes through the points $A = (0, t)$ and $D = (1, 0)$. It intersects the lines OL at B and OM at C respectively, where $L = (1, t)$ and $M = (1, 1/t)$.

and $F_2(\Lambda^*) = F(1, 1/t) = 1/\beta$ $F(\beta, \beta/t) = 1/\beta$.

Let $G(x_1, x_2) = 1$ be the equation of the polygonal arc ABCD. Then for this distance function G ,

$$G_1(\Lambda) = 1 \text{ and } G_2(\Lambda^*) = 1/\beta$$

let $\mu_G = 4 \times \text{area} \{ \text{polygon OABCD} \}$
Then from Lemma 2 we have

$$\mu_G G_1(\Lambda) G_2(\Lambda^*) = \mu_G \cdot 1/\beta \geq 2(t + 1/t) \text{ for all suitable } \alpha \text{ and } \beta.$$

Now let $\mu = \text{area} \{ \underline{x} \in \mathbb{R}^2 : F(\underline{x}) \leq 1 \}$.

Then from the convexity $\mu \geq \mu_G$.

Hence $\mu F_1(\Lambda) F_2(\Lambda^*) = \mu \cdot 1/\beta \geq \mu_G \cdot 1/\beta \geq 2(t+1/t)$.
ie we have proved that

$\mu F_1(\Lambda) F_2(\Lambda^*) \geq 2(t+1/t)$ for all convex symmetric distance functions F and the lattice Λ with basis $(1,0)$ and $(0,t)$.

$$\therefore \min_F \mu F_1(\Lambda) F_2(\Lambda^*) \geq 2(t+1/t).$$

Now we proceed to prove the right hand side of the inequality in the theorem.

Let Λ be any two dimensional lattice and $(a,b) \in \Lambda \cap P^0$ and $(-c,d) \in \Lambda \cap Q$ be two points such that $F(a,b)$ and $F(d,c)$ are minimal, where Q is the open second quadrant.

Note that if $(-c,d) \in \Lambda$, then $\frac{(d,c)}{d(\Lambda)} \in \Lambda^*$, where $d(\Lambda)$ is the determinant of Λ .

$$\text{Let } D = \{ (x_1, x_2) \in \mathbb{R}^2 : F(x_1, x_2) \leq F(a,b), F(-x_2, x_1) \leq F(d,c) \}.$$

Then from the choice of (a,b) and $(-c,d)$ there are non-zero points of Λ in D .

Then there are no non-zero points of Λ in the parallelogram.

$$\{ (x_1, x_2) \in \mathbb{R}^2 : | -x_1 + tx_2 | \leq tF(d,c), | tx_1 + x_2 | \leq tF(a,b) \}$$

which lies entirely in D .

Hence by Minkowski's linear form theorem¹, we have

$$t^2 F(a,b) F(d,c) \leq (1+t^2) d(\Lambda).$$

$$\therefore F(a,b) \frac{F(d,c)}{d(\Lambda)} \leq 1 + 1/t^2.$$

$$\text{Now } \mu = \text{area}\{(x_1, x_2) \in \mathbb{R}^2 : F(x_1, x_2) \leq 1\}$$

$\leq 4t$, as $F(x_1, x_2)$ is convex and symmetric.

$$\therefore \mu F_1(\Lambda) F_2(\Lambda^*) \leq 4(t+1/t).$$

References:

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3. HOSSAIN, F. & WORLEY, R. T., '*Positive Points in Polar Lattices*' (Personal Communication).