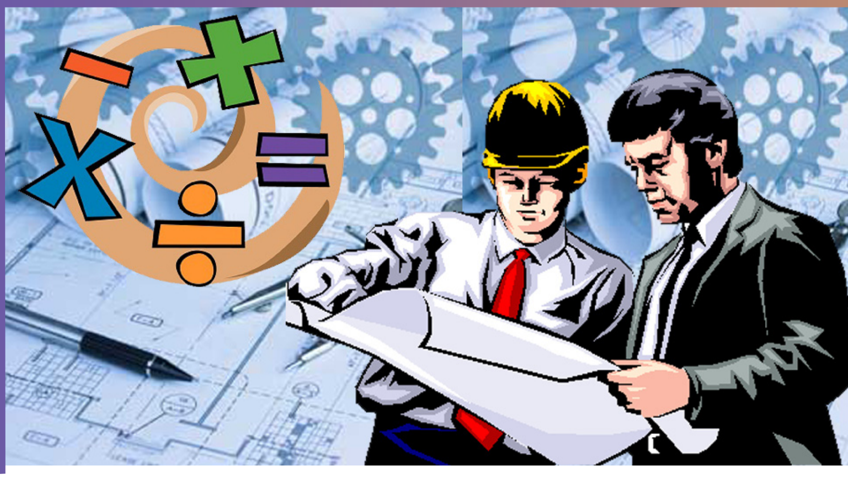
Mathematics in Engineering



Prof. Priyan Dias

Introduction: Is Engineering "Mathematics in Disguise"?

Engineering is sometimes called "Mathematics in disguise", judging from what is taught in most engineering faculties. Such faculties in most parts of the world also want potential entrants to demonstrate good skills in mathematics at secondary school level. Nevertheless, an engineer is very different from a pure mathematician, and some of these differences are also explored in this article.

So what are the kinds of mathematics used by engineers? In fact a wide variety is used, ranging from simple linear equations to calculus and statistics. Someone has defined engineering as "the study of responses of objects and systems to perturbations, in the time, space and frequency domains."

Mechanics

The mechanics based engineering disciplines (such as civil and mechanical engineering) rely heavily on Newton's Laws, whether to describe equilibrium or motion. So, although a body is in equilibrium only if at rest or moving with uniform velocity, an equilibrium equation can be written even for a mass (m) with acceleration (a), by considering the term "ma" as a force, giving rise for example to the equation describing simple harmonic motion:

$$ma = -kx,$$

where k is a spring constant, x the displacement from an equilibrium position and the term "kx" a restoring force (see Figure 1).

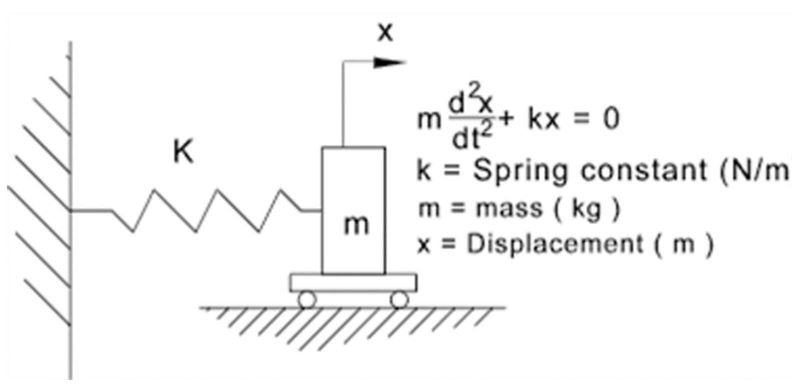


Figure 1 – The mathematics of simple harmonic motion

A lot of engineering has to deal with non-linear behaviour. For this reason, differential equations are the "bread and butter" of engineers. In the simple harmonic equation described above, the acceleration is not constant, but changes with time. Hence, we write acceleration as dv/dt (or change in velocity within an infinitesimally small time dt), and similarly velocity as dx/dt. This will cause the above simple harmonic motion equation to be written as

$$m (d2x/dt2) = -kx,$$

and this is called a differential equation, which if solved will give us an expression for how x changes with t.

In the above problem, we considered only a point mass. However, when we need to solve problems involving

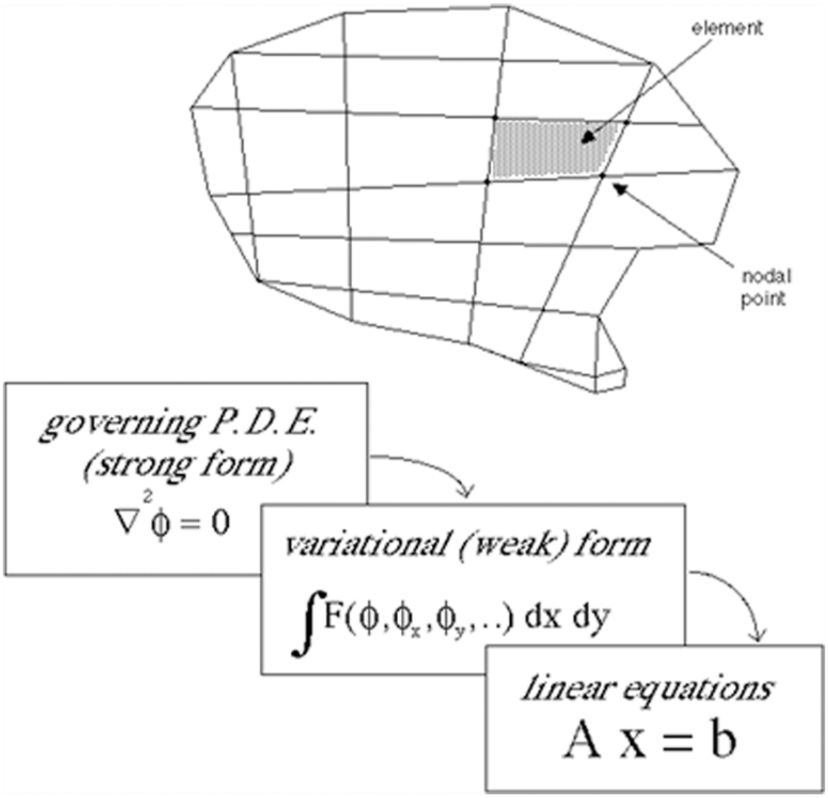


Figure 2 – Converting partial differential equations to linear simultaneous equations via finite elements

objects with complex shapes or many parts, we often resort to the finite element method. Here the object is divided (or "meshed") into small "finite elements" (see Figure 2), within which displacements can be represented by simple functions that approximate the actual displacement field. We can then get expressions for strains from displacements, link stresses to strains using known elastic properties (such as Hooke's Law and Poisson's ratio), and relate the stresses to the forces applied. This way, we can find the displacements caused by a system of forces on a complex body. All the differential equations involved in this procedure are in fact converted to a set of equivalent simultaneous equations. Such simultaneous equations are solved using matrix methods, in particular matrix inversion.

"Electrics"

The notion of fields is very important in electrical engineering. For example Maxwell's equations describe electrical and magnetic fields and the relation between them. The electric motor and electricity generator are common applications of this relation between electric and magnetic fields.

Complex numbers is another branch of mathematics that is used in electrical engineering. Complex numbers have both real and imaginary components (see Figure 3), and these components are orthogonal (i.e. perpendicular) to each other. This is useful in describing the impedance of an alternating current (ac) electric circuit, which can have resistance, impedance and capacitive components. Resistance is considered the real component and the other two combined (called reactance) the imaginary one. The complex number mathematics helps us to understand why the reactance components cause a phase lag between voltage and current.

Most of the engineering entities described above (whether forces on masses or voltages in ac circuits) have direction and are hence vectors. For that reason the mathematics of vectors, and concepts such as vector products and direction cosines are used frequently in engineering. The cross product of two vectors is another vector that is at right angles to both the original vectors (see Figure 4). It can be used to describe the Lorentz force experienced by a moving electrical charge in a magnetic field (referred to above). The dot product of two vectors is a scalar, as in the case of the work done by a force vector in

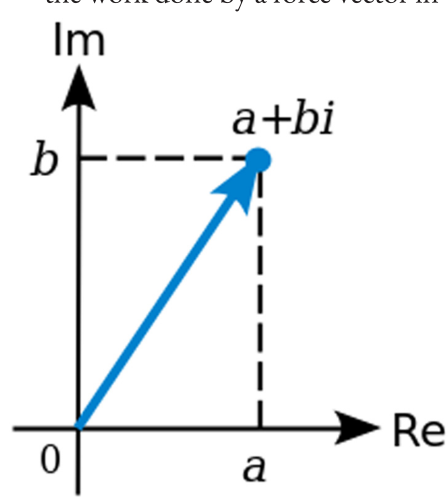


Figure 3 – Complex numbers have both Real (Re) and Imaginary (Im) components

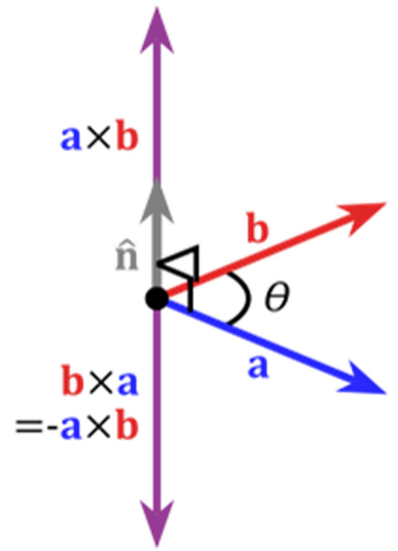


Figure 4 – The vector cross product is another vector

conjunction with a displacement vector.

Statistics

Among other things, statistics teaches us that entities can have not merely values but distributions of those values about a mean. This captures the notion of randomness, which is one aspect of uncertainty. Since engineering is carried out in an uncertain world, the mathematics of uncertainty is of great use in the practice of engineering. Figure 5 shows two normal distributions, characterized by means (μ) and standard deviations, for a load (Q) on say a structural element with resistance (R) of that element. The nature of these curves is that their "tails" are only asymptotic to the horizontal axis. There can always be a load higher than one encountered before, or a resistance lower. However, in order for structural design to be economical, we cannot design the element in a way that the highest possible load is less than the lowest possible resistance. We allow a certain overlap that is governed by an allowable probability of failure; and statistics can help us to compute this, using probability density functions such as those in Figure 5. For structural design this probability of failure is of the order of 1 in 106.

Statistics is also used in quality control, and is very important for any kind of production engineering. It can tell us how frequently tests need to be carried out and whether an unusual test result falls within the acceptable tolerance or outside it. It can guide us in making judgments about the entire population of data from results obtained for very limited samples of that population.

Design

We have hinted above about the idea of design, which is central to engineering. Design is a synthetic exercise, meaning that we try to create something, given certain requirements. Synthesis is in some ways the opposite of analysis, which involves investigating known entities. If we continue with the example of load and resistance in structural elements, the analysis task would be to find the stresses or deflections in the element, if given the loads on the element and the properties that contribute to its resistance. The synthesis task on the other hand would be to propose the properties of the element (e.g. type of material, dimensions of cross section) in order to carry the loads without exceeding certain stress and deflection limits. This is a much more difficult problem, sometimes called an "inverse" problem. Such inverse problems do not have unique solutions (unlike analysis problems). There can be more than one feasible solution. As such design can be considereda part of a wider activity called "search", i.e. search for the best solution.

Optimization problems are another type of search problem. Very often such problems are cast as minimization problems, because we seek minimum cost or minimum weight designs. Such problems also involve constraints that confine the solution to a given combination of parameters (or variables). Figure 6 is a representation of a linear function, f(x,y) that has to be minimized subject to linear constraints. The shaded area denotes the region that is allowed by the constraints and the minimum value of the function is given by one of the vertices of the triangular feasible region.

Practice

Although mathematics is so clearly central to engineering, it is only one among many tools used by an engineer. Furthermore, engineers

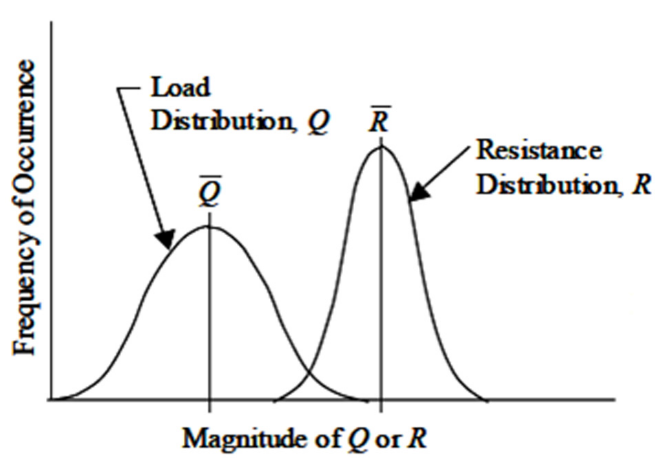


Figure 5 – Probability density functions for load and resistance

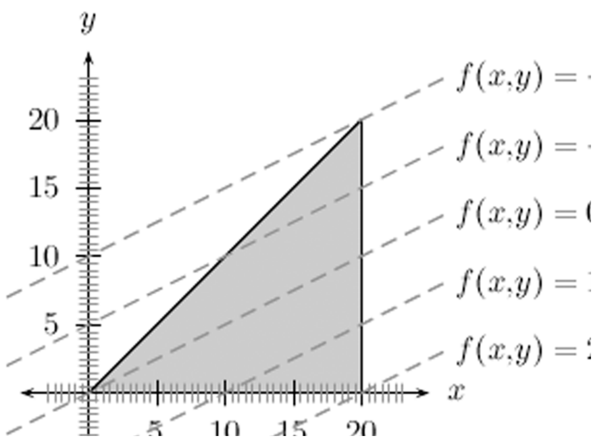


Figure 6 – Representation of a linear function subject to linear constraints

use mathematics in a very practical way, as opposed to a theoretical one. There is a story about a mathematician and an engineer who were placed at one end of a small room, with a cake at the other end. They are told that they can cross the room and take the cake. However, after one step is taken to cross the room, the next step to be taken can only be half as long as the first one, the third step half as long as the second and so on. The mathematician thinks about this and discovers that he can (theoretically) never reach the cake. The engineer however takes a few steps, gets close to the cake (because the room is small anyway), and reaches out to take the cake saying "close enough for all practical purposes". So engineers use clever approximations to arrive at practical solutions.

This need for approximation arises from what is called "idealization" in engineering. Engineers deal with a large number of objects and systems in the real world. These can range from skyscrapers and spacecraft through the network of wires that bring us electricity to mobile phones and digital cameras. It is said that humans are perhaps the only animals who build things twice. The first time round, they build things in their minds – this is probably called planning in the everyday world. During this stage, all the problems that can affect the final product are thought of and dealt with in our minds. It is only after this that the object or system is created in the real world. The "planning" done by engineers is sometimes called "design" and often called "modelling". Mathematics is very prominent in such models, which nowadays are almost always computer based. In constructing such models however, not all aspects of the real world object or problem can be represented mathematically, and approximations have to be made. This is what is called idealization. Idealization is done by erring on the side of caution, hence engineering models are said to be conservative. Accuracy is not as important for engineers as safety. It is better to have a somewhat inaccurate model that will result in a safe design, rather than a more accurate model with a lower margin of safety. These are the kinds of decisions that engineers have to make about their mathematical models.

Conclusion: Mathematics is only a part of Engineering

Mathematical models are not the only basis for design. One reason for this is that engineering artifacts have been designed and built very much before the use of mathematics in engineering. Egypt's pyramids, Rome's bridges and Sri Lanka's irrigation systems are some examples. This body of knowledge that has arisen from practice is often codified as "thumb rules". Engineers therefore will sometimes use these thumb rules to arrive at a preliminary design before using mathematical tools to refine it.

Another reason is that there are some aspects of a design that cannot be quantified, but may still be very important. For example, it may be possible, using mathematical tools, to re-design the trace of a roadway so that high speeds and reduced travel time can be achieved. However, if it is not possible to relocate the retail shops along that road, it may be that those high speeds will be unsafe for the large number of pedestrians that will be using the sidewalks. This is why Einstein is supposed to have said that "not everything that is countable counts; and not everything that counts is countable." Thinking in this way is called "reflective practice" or "systems thinking", something that all professionals are called upon to do. This is especially important for engineers, who will often be tempted to pay "selective inattention" to any aspect of a problem that cannot be represented in a mathematical model.

Prof. Priyan Dias
Department of Civil Engineering
University of Moratuwa