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ASYMPTOTIC NORMALITY OF A PROPOSED ESTIMATOR

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ABSTRACT

In a previous paper Thewarapperuma (1998) an estimator was proposed for the functional $T = \int \Phi(x)\ H(F(x))dx = \int f^2(x)dH(F(x))$. This estimator was estimated from natural considerations and its consistency was established there. In this paper it is shown that the estimator meets the other important criterion,namely, the asymptotic normality.

1. INTRODUCTION

In Thewarapperuma (1998) the consistency of the estimator $T_n = \int f_n^2(F_n^{-1}(x))dH(x)$ was established for the functional T. The function H is such that

(a) H is nondecreasing on [0,1] and
(b) It's a.e. derivative h exists but ned not be bounded.

The following assumptions were made on the function H and its derivative h.

(1) $\int dH(x) \sim C \ln(n)$ for large n, $\alpha > 0$ and C is a constant independent of n.

(2) $0 < h(x) < n^{2\alpha}$ for $x \varepsilon [n^{-\alpha}, 1- n^{-\alpha}]$.

The kernel K of the estimator was such that

(3) K is a symmetric probability density on $\mathbf{R}$ and is of bounded variation.

(4) $K^{(1)}$, the derivative of K is a continuous function of bounded variation.

(5) Window width $a_n \to 0$ and $n^{1/4}(\ln n)^{-1/2} \to \infty$ as $n \to \infty$.

Additional assumptions are needed to prove that the estimator is asymptotically normal. These assumptions and the main result are stated next.

2. ASSUMPTIONS

Let $\rho = fF^{-1}$ and suppose that

(6) $\int_0^1 | \rho^1(t-s) - \rho^1(t) | \rho(t)\, dH(t) \to 0$ as $s \to 0$

(7) $\int_0^1 f^4(F^{-1}(t)\{h(t)\}^2\,dt < C_3 < \infty$ and

(8) $f(x)\,h(F(x)) < M < \infty$ for some constant M.

3. MAIN RESULT

Theorem

Under assumptions in section 2

$$n^{1/2}\int |f_n^2(F_n^{-1}(t)) - f^2(F^{-1}(t))|\,dH(t)$$

converges in distribution to a normal random variable with mean 0 and variance σ^2 where

$$\sigma^2 = 4\int_0 f^4(F^{-1}(t))\{h(t)\}^2\,dt + 4\int[\,{}_{F(u)}\int^1 (F^{-1}(t)\,dH(t)\,]^2 f(u)du$$
$$-8\int|\,{}_{F(u)}\int^1 f'(F^{-1}(t)\,dH(t))\,|\,f^3(u)\,h(F(u))\,du$$
$$-4\,[\,{}_0\int^1 f^2(F^{-1}(t))\,dH(t) - {}_0\int^1 tf'(F^{-1}(t))\,dH(t)\,]^2$$

Proof:

Consider the integral $2n^{1/2}\int\{f_n(F^{-1}(t)) - f(F^{-1}(t))\}\,f(F^{-1}(t)\,dH(t)$

$$= 2n^{1/2}\int\{f_n(F_n^{-1}(t)) - f_n(F^{-1}(t))\}f(F^{-1}(t)\,dH(t)$$
$$+ 2n^{1/2}\int\{f_n(F^{-1}(t)) - f(F^{-1}(t))\}\,f(F^{-1}(t))\,dH(t)$$
$$= I_{n1} + I_{n2} \text{ (say).}$$

Now $I_{n1} = 2n^{1/2}.\int f(F^{-1}(t)\{\rho_n(FF_n^{-1}(t)) - \rho_n(t)\}dH(t)$; where $\rho_n = f_n F^{-1}$

$$= 2n^{1/2}.\int f(F^{-1}(t))(FF_n^{-1}(t) - t)\,\rho'_n(t + \theta_2(FF_n^{-1}(t) - t)\,dH(t);\quad 0<\theta_2<1.$$

Next it is shown that ρ'_n appearing in the above expression can be approximated by ρ.

Let $u = t + \theta_2(FF_n^{-1}(t) - t)$. Then

$$|\,2n^{1/2}\int f(F^{-1}(t))(FF_n^{-1}(t) - t)\{\,|\rho_n'(u) - \rho'(u)\}\,dH(t)\,|$$
$$\le 2n^{1/2}\,\|FF_n^{-1} - I\|\,\|(\rho_n' - \rho')\rho\|\,C\ln(n)$$
$$= 2n^{1/2}\,C\,\|FF_n^{-1} - I\|\,\|f_n' - f'\|\ln(n)$$

Setting $r = 1$ and $\varepsilon_n = \varepsilon\ln(n)$, where $\varepsilon > 0$, in Schuster's lemma (1969) we see that

$\|f_n' - f'\|\ln(n) \to 0$ in probability as $n \to \infty$ provided $n^{1/2}(\ln n)^{-1/2} a_n \to \infty$ as $n \to \infty$.

Assumption (6) implies that the term

$$|2n^{1/2} \int f(F^{-1}(t)\,(FF_n^{-1}(t) - t)\{\rho'(u) - \rho'(t)\}dH(t)| \to 0 \text{ as } n \to \infty.$$

These reductions imply that it is sufficient to show the asymptotic normality of

$$2n^{1/2} \int f(F^{-1}(t))\{FF_n^{-1}(t) - t\}[f'(F^{-1}(t) / f(F^{-1}(t)]\,dH(t)$$

$$+ 2n^{1/2} \int f(F^{-1}(t)\{f_n(F^{-1}(t) - f(F^{-1}(t)\}dH(t)$$

$$= 2n^{1/2} \int f'(F^{-1}(t))\{FF_n^{-1}(t) - t\}dH(t) + 2n^{1/2} \int f(F^{-1}(t)\{f_n(F^{-1}(t) - f(F^{-1}(t)\}dH(t)$$

Note that

$$FF_n^{-1}(t) - t = -\{F_nF^{-1}FF_n^{-1}(t) - FF_n^{-1}(t)\} + F_nF_n^{-1}(t) - t.$$

Since $\| F_nF_n^{-1} - I \| = 0_p(1/n)$ (Koul) and

$n^{1/2}(F_nF^{-1}FF_n^{-1}(t) - FF_n^{-1}(t)) \approx n^{1/2}(F_nF^{-1}(t) - t)$ it suffices to consider

$$2n^{1/2}\,[\int f(F^{-1}(t)\{f_n(F^{-1}(t)) - f(F^{-1}(t))\}dH(t) - \int f'(F^{-1}(t))\{F_nF^{-1}(t) - t\}\,dH(t)]$$

$$= 2n^{1/2}(1/n) \sum_{i=1}^{n} \{\int f(F^{-1}(t))(1/a_n)K([F^{-1}(t) - X_i]/a_n)\,dH(t)$$

$$- \int f'(F^{-1}(t))\,I_{(X_i \le F^{-1}(t))}\,dH(t) - \int f^2(F^{-1}(t))dH(t) + \int f'(F^{-1}(t))\,t\,dH(t)\}$$

$$= (1/\sqrt{n}) \sum_{i=1}^{n} (Y_{in} - A_n) \text{ ,say.}$$

Then Y_{in} are I. I. D. for each n and it is easy to see that $E(Y_{in} - A_n) \to 0$ as $n \to \infty$.

Note that

$$E(Y_{in}^2) = \int (1/a_n) \int f(F^{-1}(t)\;K([F^{-1}(t) - v]/a_n)\,dH(t)\,f(v)dv$$

$$+ \int \{\int f'(F^{-1}(t))\,I_{(v \le F^{-1}(t))}\,dH(t)\}^2 f(v)\,dv$$

$$- 2\int\{(1/a_n) \int f(F^{-1}(t)\;K([F^{-1}(t) - v]/a_n)\,dH(t) \cdot \tag{9}$$

$$\int f'(F^{-1}(s))\,I_{(v \le F^{-1}(s))}\,dH(s)\}\,f(v)dv.$$

From (8) it follows that

$$|(1/a_n) \int f(F^{-1}(t)\;K([F^{-1}(t) - v]/a_n)\,dH(t)| \le M.$$

Since $1/a_n \int f(F^{-1}(t)\;K([F^{-1}(t) - v]/a_n)\,dH(t) \to f^2(v)\,h(F(v))$

by dominated convergence theorem it follows that the first term in (9) converges to

$$\int f^{5}(x)\{h(F(x)\}^{2}dx.$$

The second and the third terms clearly converge to

$$\int [\ _{F(u)}\int (F^{-1}(t)\,dH(t)\,]^{2}\ f(u)du \quad \text{and} \quad -2\int |\ _{F(u)}\int\ f^{\prime}(F^{-1}(t)\,dH(t))\ |\ f^{3}(u)\,h\,(F(u))\,du$$

respectively. Since $E(Y_{1n})$ converges to $_0\int^1 f^{2}(F^{-1}(t))\,dH(t)\ -\ _0\int^1 tf^{\prime}(F^{-1}(t))\,dH(t)\,]^{2}$ the limiting variance is given by σ^{2}.

Remark: The assumptions made here are reasonable and can be verified for the distributions for which the integral T is finite. Normal, double exponential, logistic and Cauchy distributions are some examples.

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REFERENCES

1. Koul H.L., *Weighted Empiricals and Linear Models Vol.* **21** IMS publications
2. Schuster E., *Ann. Math. Stat.* **40 b** 1187-1195 (1969)
3. Thewarapperuma P.S.S.,*Ceylon Journal of Science:PS 5(1), 81-84 (1998)*