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RESEARCH ARTICLE

A study of obstacle problems using homotopy perturbation method

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Abstract: Numerical methods for solving differential equations has become an important topic of this era. The importance of boundary value problems in applied sciences shows the way in which existence of exact solution is not always possible. This study adopts the homotopy perturbation method (HPM) to solve multiple-point boundary value problems arising in obstacle, unilateral and contact problems. Convergent approximate solutions are constructed such that the exact boundary conditions are satisfied. Some examples have been presented to elucidate the efficiency and implementation of the method. We have compared the results using different number of terms of HPM and found that increasing the number of terms of approximate solution will increase the efficiency.

Keywords: Homotopy perturbation method, numerical solution, obstacle problem, system of boundary value problem.

INTRODUCTION

Homotopy perturbation method (HPM) is presented to approximate the solutions of multiple order obstacle, unilateral and contact problems described in its general form as:

$$u^{(n)}(x) = \begin{cases} P(f(x), g(x), u(x), r), & a \leq x \leq c \\ Q(f(x), g(x), u(x), r), & c \leq x \leq d \\ R(f(x), g(x), u(x), r), & d \leq x \leq b \end{cases} \qquad \text{...(1)}$$

With the associated boundary conditions depending on the value of n and given as

$$\begin{aligned} &u^{(k)}(a) = u^{(k)}(b) = \alpha_k, \quad k = 0,1,2,\dots,n-1 \\ &u^{(k)}(c) = \beta_k, u^{(k)}(d) = \gamma_k, \quad k = 0,1,2,\dots,n-1 \end{aligned} \qquad \text{...(2)}$$

The functions *P, Q* and *R* are from $\mathbb{R}^4 \to \mathbb{R}$. It is given that the functions

$$u^{(k)}(x), \quad k = 0,1,2,\dots,n-1$$

are continuous on c and d. The parameters α_k, β_k and $\gamma_k, k = 0,1,2,\dots,n-1$ are real constants (some finite values). Generally it is impossible to acquire the analytical form of the solution of equation (1) for arbitrary choice of $f(x)$ and $g(x)$. For this purpose some numerical methods are opted to get approximate solutions of the problems similar to equation (1). Such type of systems arise in the study of obstacle, unilateral and contact boundary value problems and have important applications in other branches of pure and applied sciences. Second order obstacle problem is solved in 2001 using the cubic spline technique (Al-said, 2001). The same second order problem has also been solved using the parametric cubic spline approach (Khan & Aziz, 2003), cubic lagrange polynomials (Iqbal, 2010) and Galerkin's finite element method (FEM) (Iqbal *et al*., 2010). In 2011 B-spline technique was used to solve the same system of second order boundary value problem (Loghmani *et al*., 2011). In 2013 adoptive FEM technique was used to address the same (Iqbal *et al*., 2013).

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Third order obstacle problem was solved using finite difference method (Noor & Al-Said, 2002) in 2002, cubic spline method (Al-said & Noor, 2003) in 2003 and variational iteration method (Geng & Cui, 2010). The HPM was also successfully applied to find the approximate solution of phase Stefan problem with variables latent heat (Rajeev, 2014).

The solution of the fourth order system of boundary value problem was presented in 2007 using the non polynomial spline technique (Siddiqi & Akram, 2007a) and cubic non polynomial spline technique (Siddiqi & Akram, 2007b). Also a solution of fourth order system was suggested using Galerkin's finite element method in 2011 (Iqbal, 2011).

In this paper, homotopy perturbation method is used to solve systems of boundary value problems of different orders. HPM, proposed by He (1999; 2003; 2006; 2009), for solving differential and integral equations is the subject of extensive analytical and numerical studies. Also the HPM is described for the nonlinear system of boundary value problems by He (2014a; 2014b). The HPM is applied to solve different engineering and applied sciences natural phenomena like fluid flow problems (Siddiqui *et al*., 2006; Ghori *et al*., 2007).

Homotopy perturbation method (an algorithm)

The formulation of the working algorithm of the homotopy perturbation method can be expressed in the following way, write the governing differential equation

$$A(u(x)) + h(x) = 0, x \in \Omega \qquad ...(3)$$

with the boundary conditions

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, r \in \Gamma \qquad ...(4)$$

where Γ is the boundary of Ω, which is the domain of definition of the following governing differential equation and $h(x)$ is the known analytic forcing function.

Equation (3) is decomposed into $A(u) = L(u) + N(u)$, where $L(u)$ is the linear part and $N(u)$ is a nonlinear part. However, these notations are not necessarily fixed. Therefore, one has the freedom to consider a linear part $L(u)$ from the governing equation (3).

(a) Construct the He's homotopy, mean to find $\varphi(x;p) : \Omega \times [0,1] \rightarrow \mathbb{R}$, which satisfies

$$\mathcal{H}(\phi(x;p) = (1-p)(L(\phi(x;p) - L(u_0)) + p\left(A(u(x;p)) + h(x)\right) = 0 \qquad ...(5)$$

or

$$\mathcal{H}(\phi(x;p) = L(\phi(x;p) - L(u_0) + pL(u_0)) + p(N(\phi(x;p)) + h(x)) = 0 \qquad ...(6)$$

where $p \in (0,1)$ is an embedding parameter. The function u_0 is an initial guess, which satisfies the boundary conditions. Obviously from equations (5) and (6), one has

$$\mathcal{H}(\phi(x;0) = L(\phi(x;0)) - L(u_0) = 0 \qquad ...(7)$$

$$\mathcal{H}(\phi(x;1) = A(\phi(x;1)) + h(x) = 0 \qquad ...(8)$$

The changing process of p from 0 to 1 is just of $\phi(x;p)$ from $u_0(x)$ to $u(x)$.

(b) The solution of equations (5) and (6) can be expressed as a power series in p:

$$\phi(x;p) = \sum_{i \geq 0} p^i \phi_i(x) \qquad ...(9)$$

As $p \rightarrow 1$, convergence of the series given in equation (9) has been observed and the result gives the approximate solution of the governing equation (3),

$$u(x) = \lim_{p \rightarrow 1} \phi(x;p) = \phi_0(x) + \phi_1(x) + \phi_2(x) \ldots \qquad ...(10)$$

METHODOLOGY

Solution of second order obstacle problems using homotopy perturbation method

In this section, the solution of some second order obstacle problems is given using homotopy perturbation method.

Example 3.1. We consider the obstacle problem of the following form

$$u'' = \begin{cases} 0 & for\ 0 \leq x < \frac{\pi}{4}\ and\ \frac{3\pi}{4} < x \leq \pi \\ u - 1 & for\ \frac{\pi}{4} \leq x \leq \frac{3\pi}{4} \end{cases} \qquad ...(11)$$

with boundary conditions $u(0) = u(\pi) = 0$. The analytical solution for the above mentioned example is given by

Al-said (2001), Khan & Aziz (2003), Iqbal (2010), Iqbal *et al*. (2010), Loghmani *et al*. (2011) and Iqbal *et al*. (2013).

$$u(x) = \begin{cases} \dfrac{4x}{\pi+4coth\frac{\pi}{4}} & for\ \ 0 \le x < \dfrac{\pi}{4} \\ 1 - \dfrac{4cosh\left(\frac{\pi}{2}-x\right)}{\pi sinh\frac{\pi}{4}+4cosh\frac{\pi}{4}} & for\ \ \dfrac{\pi}{4} \le x \le \dfrac{3\pi}{4} \\ \dfrac{4(\pi-x)}{\pi+4coth\frac{\pi}{4}} & for\ \ \dfrac{3\pi}{4} < x \le \pi \end{cases} \qquad ...(12)$$

The problem is divided into three cases and in each case it is solved up to 5 terms and $L(u_0) = 0$ (initial guess). Let $\phi(x;p) = \phi_0(x) + p\phi_1(x) + p^2\phi_2(x) + p^3\phi_3(x) + + p^4\phi_4(x) + p^5\phi_5(x)$ be the solution of the equation (11).

Case 1 $\left(0 \le x \le \frac{\pi}{4}\right)$

According to the HPM

$$L\big(\phi(x;p)\big) = \frac{d^2\phi(x;p)}{dx^2},$$

$$N\big(\phi(x;p)\big) = 0$$

By means of HPM a series of problems is generated

$$\frac{d^2\phi_0(x)}{dx^2} = 0,\ \phi_0(0) = 0,\ \phi_0\left(\frac{\pi}{4}\right) = c \qquad ...(13)$$

$$\frac{d^2\phi_i(x)}{dx^2} = 0,\ \phi_i(0) = 0,\ \phi_i\left(\frac{\pi}{4}\right) = 0 \qquad ...(14)$$

where $i = 1,2,...,5$ and c is a constant.

Now ϕ_0 and ϕ_i's can be found by solving equations (13) and (14). In this manner we will construct the solution for $\phi(x; p)$ that allows p → 1 to obtain $u_{\left[0,\frac{\pi}{4}-\right)}(x)$.

The zeroth order solution from equation (13), we get $\phi_0(x) = \frac{4cx}{\pi}$ and from equation (14) we obtain $\phi_i(x) = 0$ for $i = 1,2,...,5$.

Now by $u_{\left[0,\frac{\pi}{4}-\right)}(x) = \phi_0(x) + \phi_1(x) + \phi_2(x) + \phi_3(x) + \phi_4(x) + \phi_5(x)$ we get

$$u_{\left[0,\frac{\pi}{4}-\right)}(x) = \frac{4cx}{\pi} \qquad ...(15)$$

Case 2 $\left(\frac{3\pi}{4} \le x \le \pi\right)$

Exactly, in the same way as in case 1 we get

$$u_{\left(\frac{3\pi}{4},\pi-\right]}(x) = \frac{4b(\pi-x)}{\pi} \qquad ...(16)$$

where $b = \phi_0\left(\frac{3\pi}{4}\right)$ is a constant.

Case 3 $\left(\frac{\pi}{4} \le x \le \frac{3\pi}{4}\right)$

By means of HPM we generate a series of problems

$$\frac{d^2\phi_0(x)}{dx^2} = 0,\ \phi_0\left(\frac{3\pi}{4}\right) = b,\ \phi_0\left(\frac{\pi}{4}\right) = c \qquad ...(17)$$

$$\frac{d^2\phi_1(x)}{dx^2} - \phi_0(x) + 1 = 0,\ \phi_1\left(\frac{3\pi}{4}\right) = 0,\ \phi_1\left(\frac{\pi}{4}\right) = 0 \qquad ...(18)$$

$$\frac{d^2\phi_i(x)}{dx^2} - \phi_{i-1}(x) = 0,\ \phi_i\left(\frac{3\pi}{4}\right) = 0,\ \phi_i\left(\frac{\pi}{4}\right) = 0 \qquad ...(19)$$

where $i = 2, ... , 5$. Now $\phi_0(x)$, $\phi_1(x)$ and $\phi_i(x)$'s are found by solving equations (17), (18) and (19).

$$\phi_0(x) = -\frac{b\pi-3c\pi-4bx+4cx}{2\pi} \qquad ...(20)$$

$$\phi_1(x) = \frac{((-6+b+5c)\pi+4(b-c)x)(3\pi^2-16\pi x+16x^2)}{192\pi} \qquad ...(21)$$

$$\phi_2(x) = -\frac{1}{184320\pi}(3\pi^2 - 16\pi x + 16x^2)(5(-6 + 5b + c)\pi^3$$
$$+4(-120 + 31b + 89c)\pi^2 x + 48(10 + b - 11c)\pi x^2 + 192(-b + c)x^3) \qquad ...(22)$$

In this manner, the solution is constructed for $\phi(x;p)$ that allows p → 1 to obtain $u_{\left[\frac{\pi}{4},\frac{3\pi}{4}\right]}(x)$. Now by using the following conditions of continuity we find the value of b and c,

$$\lim_{x\to\frac{\pi}{4^-}} \frac{du(x)}{dx} = \lim_{x\to\frac{\pi}{4^+}} \frac{du(x)}{dx} \qquad ...(23)$$

$$\lim_{x\to\frac{\pi}{4^-}} \frac{du(x)}{dx} = \lim_{x\to\frac{\pi}{4^+}} \frac{du(x)}{dx} \qquad ...(24)$$

we have

Table 1: Comparison of absolute errors of example 3.1 at different orders of approximations

x	$u(x)_{Exact}$	Absolute error$_5$	Absolute error$_{10}$	Absolute error$_{15}$
0	0	0	0	0
$\frac{\pi}{8}$	0.16998	8.50667×10^{-5}	8.30942×10^{-8}	8.11467×10^{-11}
$\frac{\pi}{4}$	0.33996	1.70133×10^{-4}	1.66188×10^{-7}	1.62293×10^{-10}
$\frac{3\pi}{8}$	0.462792	2.54506×10^{-4}	2.48604×10^{-7}	2.42778×10^{-10}
$\frac{\pi}{2}$	0.50171	2.92537×10^{-4}	2.85754×10^{-7}	2.79056×10^{-10}
$\frac{5\pi}{8}$	0.462792	2.54506×10^{-4}	2.48604×10^{-7}	2.42778×10^{-10}
$\frac{3\pi}{4}$	0.33996	1.70133×10^{-4}	1.66188×10^{-7}	1.62293×10^{-10}
$\frac{7\pi}{8}$	0.16998	8.50667×10^{-5}	8.30942×10^{-8}	8.11467×10^{-11}
π	0	0	0	0

Absolute error$_i$, where $i = 5,10,15$ indicate that solution is obtained using HPM up to i th order.

$$b = \frac{\pi^2(92897280-1935360\pi^2+48384\pi^4-1224\pi^6+31\pi^8)}{1486356480+92897280\pi^2-1935360\pi^4+48384\pi^6-1224\pi^8+31\pi^{10}} \qquad ...(25)$$

$$c = \frac{\pi^2(92897280-1935360\pi^2+48384\pi^4-1224\pi^6+31\pi^8)}{1486356480+92897280\pi^2-1935360\pi^4+48384\pi^6-1224\pi^8+31\pi^{10}} \qquad ...(26)$$

By putting the values of b and c in $u_{\left[0,\frac{\pi}{4}\right)}(x)$, $u_{\left[\frac{\pi}{4},\frac{3\pi}{4}\right]}(x)$ and $u_{\left(\frac{3\pi}{4},\pi-\right]}(x)$, the approximate solution is obtained, which is shown in the Table 1.

Example 3.2. We consider the obstacle problem of the following form

$$u'' = \begin{cases} 2 & for\ 0 \le x < \frac{\pi}{4}\ and\ \frac{3\pi}{4} < x \le \pi \\ u+1 & for\ \frac{\pi}{4} \le x \le \frac{3\pi}{4} \end{cases} \qquad ...(27)$$

with boundary conditions $u(0) = u(\pi) = 0$. The analytical solution for the above mentioned example is given by Loghmani *et al*. (2011):

$$u(x) = \begin{cases} x^2 + \left(\frac{(\pi^2-16)sinh\frac{\pi}{4}}{4\pi sinh\frac{\pi}{4}+16cosh\frac{\pi}{4}} - \frac{\pi}{2}\right)x & for \quad 0 \le x < \frac{\pi}{4} \\ -1 - \frac{(\pi^2-16)\cosh\left(\frac{\pi}{2}-x\right)}{4\pi sinh\frac{\pi}{4}+16cosh\frac{\pi}{4}} & for \quad \frac{\pi}{4} \le x \le \frac{3\pi}{4} \\ x^2 + \left(\frac{(\pi^2-16)sinh\frac{\pi}{4}}{4\pi sinh\frac{\pi}{4}+16cosh\frac{\pi}{4}}\right)(\pi - x) + \frac{\pi}{2}(\pi - 3x) & for \quad \frac{3\pi}{4} < x \le \pi \end{cases} \qquad ...(28)$$

The problem is divided into three cases and in each case it is solved up to 5 terms and $L(u_0) = 0$ (initial guess). Let $\phi(x;p) = \phi_0(x) + p\phi_1(x) + p^2\phi_2(x) + p^3\phi_3(x) + p^4\phi_4(x) + p^5\phi_5(x)$ be the solution of the equation (27).

Case $1\left(0 \le x \le \frac{\pi}{4}\right)$

According to the HPM

$$L(\phi(x;p)) = \frac{d^2\phi(x;p)}{dx^2} - 2,$$

$$N(\phi(x;p)) = 0$$

By means of HPM a series of problems arises

$$\frac{d^2\phi_0(x)}{dx^2} - 2 = 0,\ \phi_0(0) = 0,\ \phi_0\left(\frac{\pi}{4}\right) = c \qquad ...(29)$$

$$\frac{d^2\phi_i(x)}{dx^2} = 0,\ \phi_i(0) = 0,\ \phi_i\left(\frac{\pi}{4}\right) = 0 \qquad ...(30)$$

where $i = 1,2,...,5$ and c is a constant.

Now ϕ_0 and ϕ_i´s can be found by solving equations (29) and (30). In this manner we will construct the solution for $\phi(x;\, p)$ that allows p → 1 to obtain $u_{\left[0,\frac{\pi}{4}\right)}(x)$.

The zeroth order solution from equation (29), we get $\phi_0(x) = x^2 + \frac{4cx}{\pi} - \frac{\pi x}{4}$ and from equation (30) we obtain for $i = 1,2,...,5$.

Now by $u_{\left[0,\frac{\pi}{4}\right)}(x) = \phi_0(x) + \phi_1(x) + \phi_2(x) + \phi_3(x) + \phi_4(x) + \phi_5(x)$ we get

$$u_{\left[0,\frac{\pi}{4}\right)}(x) = x^2 + \frac{4cx}{\pi} - \frac{\pi x}{4} \qquad ...(31)$$

Case $2\left(\frac{3\pi}{4} \le x \le \pi\right)$

Same way as in case 1 we get

$$u_{\left(\frac{3\pi}{4},\pi\right]}(x) = \frac{(16b+\pi(3\pi-4x))(\pi-x)}{4\pi} \qquad ...(32)$$

where $b = \phi_0\left(\frac{3\pi}{4}\right)$ is the constant.

Case $3\left(\frac{\pi}{4} \le x \le \frac{3\pi}{4}\right)$

By means of HPM

$$\frac{d^2\phi_0(x)}{dx^2} = 0,\ \phi_0\left(\frac{3\pi}{4}\right) = b,\ \phi_0\left(\frac{\pi}{4}\right) = c \qquad ...(33)$$

$$\frac{d^2\phi_1(x)}{dx^2} - \phi_0(x) - 1 = 0,\ \phi_1\left(\frac{3\pi}{4}\right) = 0,\ \phi_1\left(\frac{\pi}{4}\right) = 0 \qquad ...(34)$$

$$\frac{d^2\phi_i(x)}{dx^2} - \phi_{i-1}(x) = 0,\ \phi_i\left(\frac{3\pi}{4}\right) = 0,\ \phi_i\left(\frac{\pi}{4}\right) = 0 \qquad ...(35)$$

where $i = 2, ..., 5$. Now $\phi_0(x)$, $\phi_1(x)$ and $\phi_i(x)$´s are found by solving equations (33), (34) and (35).

$$\phi_0(x) = -\frac{b\pi-3c\pi-4bx+4cx}{2\pi} \qquad ...(36)$$

$$\phi_1(x) = \frac{((6+b+5c)\pi+4(b-c)x)(3\pi^2-16\pi x+16x^2)}{192\pi} \qquad ...(37)$$

$$\phi_2(x) = -\frac{1}{184320\pi}(3\pi^2 - 16\pi x + 16x^2)(5(6 + 5b + c)\pi^3$$
$$+4(120 + 31b + 89c)\pi^2 x + 48(-10 + b - 11c)\pi x^2 + 192(-b + c)x^3) \qquad ...(38)$$

Furthermore, the solution is constructed for $\phi(x;p)$ that allows p → 1 to obtain $u_{\left[\frac{\pi}{4},\frac{3\pi}{4}\right]}(x)$. Now by using the following conditions of continuity the value of b and c can be found,

$$\lim_{x\to\frac{\pi}{4}^-}\frac{du(x)}{dx} = \lim_{x\to\frac{\pi}{4}^+}\frac{du(x)}{dx} \qquad ...(39)$$

$$\lim_{x\to\frac{3\pi}{4}^-}\frac{du(x)}{dx} = \lim_{x\to\frac{3\pi}{4}^+}\frac{du(x)}{dx} \qquad ...(40)$$

We have

$$b = -\frac{\pi^2(185794560-1935360\pi^2+48384\pi^4-1224\pi^6+31\pi^8)}{1486356480+92897280\pi^2-1935360\pi^4+48384\pi^6-1224\pi^8+31\pi^{10}} \qquad ...(41)$$

$$c = \frac{\pi^2(185794560-1935360\pi^2+48384\pi^4-1224\pi^6+31\pi^8)}{1486356480+92897280\pi^2-1935360\pi^4+48384\pi^6-1224\pi^8+31\pi^{10}} \qquad ...(42)$$

By putting the values of b and c in $u_{\left[0,\frac{\pi}{4}\right)}(x)$, $u_{\left[\frac{\pi}{4},\frac{3\pi}{4}\right]}(x)$ and $u_{\left(\frac{3\pi}{4},\pi\right]}(x)$, the approximate solution is obtained, which is shown in Table 2.

Table 2: Comparison of absolute errors of example 3.2 at different orders of approximations.

x	$u(x)_{Exact}$	Absolute error_5	Absolute error_{10}
0	0	0	0
$\frac{\pi}{8}$	− 0.527765	3.25933×10^{-5}	3.18375×10^{-8}
$\frac{\pi}{4}$	− 0.747106	6.51866×10^{-5}	6.3675×10^{-8}
$\frac{3\pi}{8}$	− 0.794169	9.75137×10^{-5}	9.52527×10^{-8}
$\frac{\pi}{2}$	− 0.80908	1.12085×10^{-4}	1.09486×10^{-7}
$\frac{5\pi}{8}$	− 0.794169	9.75137×10^{-5}	9.52527×10^{-8}
$\frac{3\pi}{4}$	− 0.747106	6.51866×10^{-5}	6.3675×10^{-8}
$\frac{7\pi}{8}$	− 0.527765	3.25933×10^{-5}	3.18375×10^{-8}
π	0	0	0

Absolute error_i, where $i = 5,10$ indicate that solution is obtained using HPM up to i^{th} order.

Solution of third order obstacle problem using homotopy perturbation method

In this section, the solution of a third order obstacle problem is given using the homotopy perturbation method.

Example 4.1. We consider the obstacle problem of the following form

$$u''' = \begin{cases} 0 & for\ 0 \le x < \frac{1}{4}\ and\ \frac{3}{4} < x \le 1 \\ u - 1 & for\ \frac{1}{4} \le x \le \frac{3}{4} \end{cases} \qquad ...(43)$$

with boundary conditions $u(0) = u'(0) = u'(1) = 0$. The analytical solution for the above mentioned example is given by Noor & Al-said (2002), Al-said & Noor (2003), Geng & Cui (2010) and Noor *et al.* (2011);

$$u(x) = \begin{cases} \frac{1}{2}a_1x^2 & for \quad 0 \le x < \frac{1}{4} \\ 1 + 2a_2e^x + e^{\frac{\pi}{2}}\left(a_3cos\frac{\sqrt{3}}{2}x + a_4sin\frac{\sqrt{3}}{2}x\right) & for \quad \frac{1}{4} \le x \le \frac{3}{4} \\ \frac{1}{2}a_5x(x-2) + a_6 & for \quad \frac{3}{4} < x \le 1 \end{cases} \qquad ...(44)$$

The approximate values of constants a_i's where i = 1,2,...,6 are

$$a_1 = 0.24391096222647513$$

$$a_2 = -0.1784723445274597$$

$$a_3 = -0.8189357356561506$$

$$a_4 = 0.24213890868443372$$

$$a_5 = -0.0653763009211085$$

The problem is divided into three cases and in each case it is solved up to 5 terms and $L(u_0) = 0$ (initial guess). Let $\phi(x;p) = \phi_0(x) + p\phi_1(x) + p^2\phi_2(x) + p^3\phi_3(x) + p^4\phi_4(x)$ $p^5\phi_5(x)$ be the solution of the equation (43).

Case 1$\left(0 \le x \le \frac{1}{4}\right)$

According to the HPM

$$L(\phi(x;p)) = \frac{d^3\phi(x;p)}{dx^3},$$

$$N(\phi(x;p)) = 0$$

By means of HPM we generate a series of problems

$$\frac{d^3\phi_0(x)}{dx^3} = 0,\ \phi_0(0) = 0, \phi_0'(0) = 0, \phi_0'\left(\frac{1}{4}\right) = a \qquad ...(45)$$

$$\frac{d^3\phi_i(x)}{dx^3} = 0,\ \phi_i(0) = 0, \phi_i'(0) = 0,\ \phi_i\left(\frac{1}{4}\right) = 0 \qquad ...(46)$$

where i = 1,2,...,5 and a is a constant.

Now ϕ_0 and ϕ_i′s can be found by solving equations (45) and (46). Moreover, we will construct the solution for $\phi(x;p)$ that allows p → 1 to obtain $u_{\left[0,\frac{1}{4}\right)}(x)$.

The zeroth order solution from equation (45), we get $\phi_0(x) = 2\alpha x^2$ and from equation (46) we obtain $\phi_i(x) = 0$ for i = 1,2,...,5.

Now by $u_{\left[0,\frac{1}{4}\right)}(x) = \phi_0(x) + \phi_1(x) + \phi_2(x) + \phi_3(x) + \phi_4(x)$ $+ \phi_5(x)$ we get

$$u_{\left[0,\frac{1}{4}\right)}(x) = 2ax^2 \qquad ...(47)$$

Case 2$\left(\frac{3}{4} \le x \le 1\right)$

According to the HPM

$$L(\phi(x;p)) = \frac{d^3\phi(x;p)}{dx^3},$$

$$N(\phi(x;p)) = 0$$

By means of HPM we generate a series of problems

$$\frac{d^3\phi_0(x)}{dx^3} = 0,\ \phi_0\left(\frac{3}{4}\right) = b, \phi_0'\left(\frac{3}{4}\right) = c, \phi_0'(1) = 0 \qquad ...(48)$$

$$\frac{d^3\phi_i(x)}{dx^3} = 0,\ \phi_i\left(\frac{3}{4}\right) = b, \phi_i'\left(\frac{3}{4}\right) = c,\ \phi_i(1) = 0 \qquad ...(49)$$

where i = 1,2,...,5 and b, c are constants.

Now ϕ_0 and ϕ_i′s can be found by solving equations (48) and (49). In this manner we will construct the solution for $\phi(x;p)$ that allows p → 1 to obtain $u_{\left(\frac{3}{4},1\right]}(x)$.

The zeroth order solution from equation (48), we get $\phi_0(x) = b + c\left(-2x^2 + 4x - \frac{15}{8}\right)$ and from equation (49) we obtain $\phi_i(x) = 0$ for i = 1,2,...,5.

Now by $u_{\left(\frac{3}{4},1\right]}(x) = \phi_0(x) + \phi_1(x) + \phi_2(x) + \phi_3(x) +$ $+ \phi_4(x) + \phi_5(x)$ we get

$$u_{\left(\frac{3}{4},1\right]}(x) = b + c\left(-2x^2 + 4x - \frac{15}{8}\right) \qquad ...(50)$$

Case 3$\left(\frac{1}{4} \le x \le \frac{3}{4}\right)$

By means of HPM we generate a series of problems

$$\frac{d^3\phi_0(x)}{dx^3} = 0,\ \phi_0\left(\frac{1}{4}\right) = d, \phi_0'\left(\frac{1}{4}\right) = a, \phi_0'\left(\frac{3}{4}\right) = c \qquad ...(51)$$

$$\frac{d^3\phi_1(x)}{dx^3} - \phi_0(x) + 1 = 0,\ \phi_1\left(\frac{1}{4}\right) = 0,\ \phi_1'\left(\frac{1}{4}\right) = 0,\ \phi_1'\left(\frac{3}{4}\right) = 0 \qquad ...(52)$$

$$\frac{d^3\phi_i(x)}{dx^3} - \phi_{i-1}(x) = 0,\ \phi_i\left(\frac{1}{4}\right) = 0,\ \phi_i'\left(\frac{1}{4}\right) = 0,\ \phi_i'\left(\frac{3}{4}\right) = 0 \qquad ...(53)$$

Where $i = 2, ..., 5$. Now $\phi_0(x)$, $\phi_1(x)$ and $\phi_i(x)$′s are found by solving equations (51), (52) and (53).

$$\phi_0(x) = -\frac{1}{16}(16d + c(1-4x)^2 + a(-5+24x-16x^2)) \qquad ...(54)$$

$$\phi_1(x) = -\frac{1}{61440}(1-4x)^2(-640(-1+d)(-1+x) + c(21-12x+48x^2-64x^3) + a(49+92x-208x^2+64x^3)) \qquad ...(55)$$

$$\phi_2(x) = -\frac{1}{1321205760}(1-4x)^2(448(-1+d)(79-88x+384x^2-640x^3+256x^4) + a(5503-3688x+13328x^2-9984x^3-24320x^4+22528x^5-4096x^6) + c(2097-1368x+5616x^2-8448x^3+3840x^4-6144x^5+4096x^6)) \qquad ...(56)$$

In this manner we will construct the solution for $\phi(x;p)$ that allows p → 1 to obtain $u_{[\frac{1}{4},\frac{3}{4}]}(x)$.

Now by using the following conditions of continuity, the value of a,b,c and d are found as follows,

$$\lim_{x\to\frac{1}{4}^-}\frac{du(x)}{dx} = \lim_{x\to\frac{1}{4}^+}\frac{du(x)}{dx} \qquad ...(57)$$

$$lim_{x\to\frac{3}{4}^-}\frac{du(x)}{dx} = lim_{x\to\frac{3}{4}^+}\frac{du(x)}{dx} \qquad ...(58)$$

$$\lim_{x\to\frac{1}{4}^-}\frac{d^2u(x)}{dx^2} = \lim_{x\to\frac{1}{4}^+}\frac{d^2u(x)}{dx^2} \qquad ...(59)$$

$$lim_{x\to\frac{3}{4}^-}\frac{d^2u(x)}{dx^2} = lim_{x\to\frac{3}{4}^+}\frac{d^2u(x)}{dx^2} \qquad ...(60)$$

We have

$$a = \frac{1992669022346486852963931135853 2}{3267863000749747597476356555998 93} \qquad ...(61)$$

$$b = \frac{33727387421442925294654282947691906707769540412129671}{70080971618268890480909922971561060131255241775513600 0} \qquad ...(62)$$

$$c = \frac{909968297844134484341987831872677}{1503216980344883894839124015759507 8} \qquad ...(63)$$

$$d = \frac{498167255586621713240982783963 3}{6535726001499495194952713111997 86} \qquad ...(64)$$

By putting the values of a, b, c and d in $u_{[0,\frac{1}{4})}(x)$, $u_{[\frac{1}{4},\frac{3}{4}]}(x)$ and $u_{(\frac{3}{4},1-]}(x)$, the approximate solution is obtained, which is shown in Table 3.

Table 3: Absolute errors of example 4.1

x	$u(x)_{Exact}$	Absolute error
0	0	0
$\frac{1}{8}$	0.00190555	3.41263×10^{-15}
$\frac{1}{4}$	0.00762222	1.36505×10^{-14}
$\frac{3}{8}$	0.0168276	3.05554×10^{-14}
$\frac{1}{2}$	0.0279161	5.40748×10^{-14}
$\frac{5}{8}$	0.0389783	8.13308×10^{-14}
$\frac{3}{4}$	0.0481263	1.06665×10^{-13}
$\frac{7}{8}$	0.0538014	1.22645×10^{-13}
1	0	0

Solution of fourth order obstacle problem using homotopy perturbation method

In this section, the solution of a fourth order obstacle problem is given using the homotopy perturbation method.

Example 5.1. We consider the obstacle problem of the following form

$$u^{(4)} = \begin{cases} 1 & for -1 \le x < -\frac{1}{2} \; and \; \frac{1}{2} < x \le 1 \\ 2-4u & for \qquad -\frac{1}{2} \le x \le \frac{1}{2} \end{cases} \qquad ...(65)$$

with boundary conditions $u(-1) = u\left(-\frac{1}{2}\right) = u\left(\frac{1}{2}\right) = u(1) = 0$, $u'(-1) = u'\left(-\frac{1}{2}\right) = u'\left(\frac{1}{2}\right) = u'(1) = 0$ and with the condition of continuity of u and u' at $x = -1, -\frac{1}{2}, \frac{1}{2}, 1$. The analytical solution for the above mentioned example is given by Siddiqi & Akram (2007a; 2007b), and Iqbal (2011):

$$u(x) = \begin{cases} \frac{1}{24}x^4 + \frac{1}{8}x^3 + \frac{13}{96}x^2 + \frac{1}{16}x + \frac{1}{96} & for \quad -1 \le x < -\frac{1}{2} \\ \frac{1}{2\beta_1}\left(\beta_1 - e^{\left(\frac{1}{2}+x\right)}(\beta_2 + e\beta_3) - e^{\left(\frac{1}{2}-x\right)}(\beta_4 + e\beta_5)\right) & for \quad -\frac{1}{2} \le x \le \frac{1}{2} \\ \frac{1}{24}x^4 - \frac{1}{8}x^3 + \frac{13}{96}x^2 - \frac{1}{16}x + \frac{1}{96} & for \quad \frac{1}{2} < x \le 1 \end{cases} \qquad ...(66)$$

Where

$$\beta_1 = e^2 - 1 + 2esin(1)$$

$$\beta_2 = sin\left(\frac{1}{2}+x\right) - cos\left(\frac{1}{2}+x\right)$$

$$\beta_3 = sin\left(\frac{1}{2}-x\right) + cos\left(\frac{1}{2}-x\right)$$

$$\beta_4 = sin\left(\frac{1}{2}-x\right) - cos\left(\frac{1}{2}-x\right)$$

$$\beta_5 = sin\left(\frac{1}{2}+x\right) + cos\left(\frac{1}{2}+x\right)$$

The problem is divided into three cases and in each case it is solved up to 5 terms and $L(u_0) = x^2$ (initial guess). Let $\phi(x;p) = \phi_0(x) + p\phi_1(x) + p^2\phi_2(x) + p^3\phi_3(x) + p^4\phi_4(x) + p^5\phi_5(x)$ be the solution of the equation (65).

Case 1 $\left(-1 \le x \le -\frac{1}{2}\right)$

According to the HPM

$$L\big(\phi(x;p)\big) = \frac{d^4\phi(x;p)}{dx^4},$$

$$N\big(\phi(x;p)\big) = 0$$

By means of HPM we generate a series of problems

$$\frac{d^4\phi_0(x)}{dx^4} - x^2 = 0,\ \phi_0(-1) = 0, \phi_0\left(-\frac{1}{2}\right) = 0, \phi_0^{'}(-1) = 0,$$
$$\phi_0{'}\left(-\frac{1}{2}\right) = 0 \qquad \text{...(67)}$$

$$\frac{d^4\phi_1(x)}{dx^4} + x^2 - 1 = 0,\ \phi_1(-1) = 0, \phi_1\left(-\frac{1}{2}\right) = 0, \phi_1^{'}(-1) = 0,$$
$$\phi_1{'}\left(-\frac{1}{2}\right) = 0 \qquad \text{...(68)}$$

$$\frac{d^4\phi_i(x)}{dx^4} = 0,\ \phi_i(-1) = 0, \phi_i\left(-\frac{1}{2}\right) = 0, \phi_i^{'}(-1) = 0,$$
$$\phi_i{'}\left(-\frac{1}{2}\right) = 0 \qquad \text{...(69)}$$

where i = 1,2,...,5.

Now ϕ_0 and ϕ_i´s can be found by solving equations (67), (68) and (69). In this manner we will construct the solution for $\phi(x;p)$ that allows p → 1 to obtain $u_{\left[-1,-\frac{1}{2}\right)}(x)$.

The zeroth order solution from equations (67) and (68), we get

$$\phi_0(x) = \frac{(1+3x+2x^2)^2(23-12x+4x^2)}{5760}$$

$$\phi_1(x) = -\frac{(1+3x+2x^2)^2(-37-12x+4x^2)}{5760}$$

and from equation (69) we obtain $\phi_i(x) = 0$ for i = 2,...,5. Now by $u_{\left[-1,-\frac{1}{2}\right)}(x) = \phi_0(x) + \phi_1(x) + \phi_2(x) + \phi_3(x) + \phi_4(x) + \phi_5(x)$ we get

$$u_{\left[-1,-\frac{1}{2}\right)}(x) = \frac{1}{24}x^4 + \frac{1}{8}x^3 + \frac{13}{96}x^2 + \frac{1}{16}x + \frac{1}{96} \qquad \text{...(70)}$$

Case 2 $\left(\frac{1}{2} \le x \le 1\right)$

Exactly in the same way as in case 1 we get

$$u_{\left(\frac{1}{2},1-\right]}(x) = \frac{1}{24}x^4 - \frac{1}{8}x^3 + \frac{13}{96}x^2 - \frac{1}{16}x + \frac{1}{96} \qquad \text{...(71)}$$

Case 3 $\left(-\frac{1}{2} \le x \le \frac{1}{2}\right)$

By means of HPM we generate a series of problems

$$\frac{d^4\phi_0(x)}{dx^4} - x^2 = 0,\ \phi_0\left(-\frac{1}{2}\right) = 0, \phi_0\left(\frac{1}{2}\right) = 0, \phi_0^{'}\left(-\frac{1}{2}\right) = 0,$$
$$\phi_0{'}\left(\frac{1}{2}\right) = 0 \qquad \text{...(72)}$$

$$\frac{d^4\phi_1(x)}{dx^4} + 4\phi_0(x) + x^2 - 2 = 0,\ \phi_1\left(-\frac{1}{2}\right) = 0, \phi_1\left(\frac{1}{2}\right) = 0,$$
$$\phi_1^{'}\left(-\frac{1}{2}\right) = 0, \phi_1{'}\left(\frac{1}{2}\right) = 0 \qquad \text{...(73)}$$

$$\frac{d^4\phi_i(x)}{dx^4} + 4\phi_{i-1}(x) = 0,\ \phi_i\left(-\frac{1}{2}\right) = 0, \phi_i\left(\frac{1}{2}\right) = 0,$$
$$\phi_i^{'}\left(-\frac{1}{2}\right) = 0, \phi_i{'}\left(\frac{1}{2}\right) = 0 \qquad \text{...(74)}$$

where i = 1,2,...,5.

Now ϕ_0 and ϕ_i´s can be found by solving equations (72), (73) and (74).

$$\phi_0(x) = \frac{1-6x^2+32x^6}{11520} \qquad \text{...(75)}$$

$$\phi_1(x) = \frac{594635-4777201x^2+9675120x^4-321888x^6-256x}{116121600} \qquad \text{...(76)}$$

$$\phi_2(x) = \frac{1}{334764638208000}(-13850276395 + 118299485094x^2$$
$$- 285710224800x^4 + 153023302432x^6 - 66410023680x^8 +$$
$$736479744x) \qquad \text{...(77)}$$

In this manner we will construct the solution for $\phi(x;p)$ that allows p → 1 to obtain $u_{\left[-\frac{1}{2},\frac{1}{2}\right]}(x)$. Hence the approximate solution is obtained, which is shown in Table 4.

Table 4: Absolute errors of example 5.1

x	$u(x)_{\text{Exact}}$	Absolute error
-1	0	0
$-\frac{4}{5}$	$\frac{3}{20000}$	0
$-\frac{3}{5}$	$\frac{1}{15000}$	0
$-\frac{1}{2}$	0	0
$-\frac{3}{10}$	0.00211701	6.53262×10^{-14}
$-\frac{1}{10}$	0.0047617	1.53442×10^{-13}
$\frac{1}{10}$	0.0047617	1.53361×10^{-13}
$\frac{3}{10}$	0.00211701	6.52443×10^{-14}
$\frac{1}{2}$	0	0
$\frac{7}{10}$	$\frac{3}{20000}$	0
$\frac{9}{10}$	$\frac{1}{15000}$	0
1	0	0

CONCLUSION

In this article, homotopy perturbation method has been used for the solution of systems of different orders of boundary value problems. The applicability of HPM has also been analysed to solve systems of boundary value problems. Examples of different orders demonstrate the fact that the solutions of HPM are very accurate and are in excellent agreement with the exact solutions.

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