J.Natn.Sci.Foundation Sri Lanka 2018 46 (3): 423 - 430
DOI: http://dx.doi.org/10.4038/jnsfsr.v46i3.8494

RESEARCH ARTICLE

Implication-based fuzzy ternary subsemigroups in ternary semigroups

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Revised: 03 January 2018; Accepted: 24 January 2018

Abstract: The aim of this study is to introduce the concept of implication-based fuzzy ternary subsemigroups in ternary semigroups. Using the four implication operators, that is, Gaines-Rescher implication operator, Gödel implication operator, the contraposition of Gödel implication operator and the Luckasiewicz implication operator, the implication-based fuzzy ternary subsemigroups are considered. Furthermore, based on this novel idea, relationships between fuzzy [resp. $(\in, \in \vee q)$-fuzzy] ternary subsemigroups and implication-based fuzzy ternary subsemigroups are discussed. Finally, conditions for a fuzzy ternary subsemigroup with thresholds 0 and 0.5 to be an implication-based fuzzy ternary subsemigroups under the Luckasiewicz implication operator are provided.

Keywords: Fuzzifying ternary semigroup, fuzzy ternary subsemigroup, *t*-implication-based fuzzy ternary subsemigroup.

INTRODUCTION

After the introduction of fuzzy set by Zadeh in 1965 (Zadeh, 1965) there are many studies devoted to fuzzify the classical mathematics into fuzzy mathematics. Rosenfeld (1971) applied the concept of fuzzy set to algebra. Algebraic structure plays a prominent role in mathematics with wide ranging applications in many disciplines such as computer sciences, control engineering, information sciences, coding theory, economics and many others. The idea of quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in Pu and Liu (1980) played a vital role to generate some different types of fuzzy subgroups called (α, β)-fuzzy subgroups, introduced by Bhakat and Das (1996). In particular, $(\in, \in \vee q)$-fuzzy subgroup is an important and useful generalisation of Rosenfeld's fuzzy subgroup. As a general form of fuzzy ternary subsemigroup, Rehman and Shabir (2012; 2013) discussed fuzzy ternary subsemigroups with thresholds γ and δ in ternary semigroups and investigated related results. Many researchers worked on implication based fuzzy subalgebraic structures, but all of them focused their attention on 0.5-implication-based fuzzy subalgebraic structures (Jun *et al*., 2010; Davvaz & Khan, 2011). Naturally, question arises whether we can define the notion of *t*-implication-based fuzzy substructures for any $t \in [0, 1]$? The aim of this paper is to define and discuss *t*-implication-based fuzzy ternary subsemigroups under the four implication-operators; Gaines-Rescher implication operator, Gödel implication operator, the contraposition of Gödel implication operator and the Luckasiewicz implication operator. Based on this novel idea, relationships between fuzzy [resp. $(\in, \in \vee q)$-fuzzy] ternary subsemigroups and implication-based fuzzy ternary subsemigroups are considered. An example is provided to show that a fuzzy ternary subsemigroup with thresholds 0 and 0.5 is not an implication-based ternary subsemigroup under the Luckasiewicz implication-operator. Then conditions are considerd for a fuzzy ternary subsemigroup with thresholds 0 and 0.5 to be an implication-based fuzzy ternary subsemigroup under the Luckasiewicz implication operator.

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PRELIMINARIES

Definition 1

A ternary semigroup is an algebraic structure consisting of a nonempty set S together with a ternary operation $(a,b,c)\rightarrow [abc]$ satisfying the associative law $[[abc]xy]=[a[bcx]y]=[ab[cxy]]$ for all $a,b,c,x,y\in S$.

Example 1.

(1) Any semigroup can be made into a ternary semigroup by defining the ternary product to be $[abc]=abc$.
(2) Let $\mathbb{Z}^-$ be the set of all negative integers. Then with usual ternary multiplication, $\mathbb{Z}^-$ forms a ternary semigroup.

A nonempty subset A of a ternary semigroup S is called a ternary subsemigroup of S if $abc\in A$ for all $a,b,c\in A$.

If S is a set, then a fuzzy set in S is a function $\mu : S\rightarrow [0,1]$. A fuzzy set μ in a set S of the form

$$\mu(y):=\begin{cases} t\in(0,1] & \text{if } y=x,\\ 0 & \text{if } y\neq x,\end{cases}$$

is said to be a fuzzy point with support x and value t and is denoted by x_t.

For a fuzzy point x_t and a fuzzy set μ in a set S, Pu and Liu (1980) introduced the symbol $x_t\alpha\mu$, where $\alpha\in\{\in, q, \in\vee q, \in\wedge q\}$. To say that $x_t\in\mu$ (resp. $x_t\, q\,\mu$), we mean $\mu(x)\geq t$ (resp. $\mu(x)+t>1$), and in this case, x_t is said to belong to (resp. be quasi-coincident with) a fuzzy set μ. To say that $x_t\in\vee q\ \mu$ (resp. $x_t\in\wedge q\ \mu$), we mean $x_t\in\mu$ or $x_t\, q\,\mu$ (resp. $x_t\in\mu$ and $x_t\, q\,\mu$).

In what follows, let S denote a ternary semigroup unless otherwise specified. A fuzzy set μ in S is called a fuzzy ternary subsemigroup of S (Kar & Sarkar, 2012) if it satisfies:

$$\mu(xyz)\geq \min\{\mu(x),\ \mu(y),\ \mu(z)\} \quad ...(1)$$

for all $x,y,z\in S$.

A fuzzy set μ in S is said to be an $(\in,\in\vee q)$-fuzzy ternary subsemigroup of S (Rehman & Shabir, 2012), if it satisfies the following condition:

$$x_{t_1}\in\mu,\ y_{t_2}\in\mu,\ z_{t_3}\in\mu \text{ imply } (xyz)_{\min\{t_1,t_2,t_3\}}\in\vee q\,\mu, \quad ...(2)$$

for all $x,y,z\in S$.

A fuzzy set μ in S is called a fuzzy ternary subsemigroup with thresholds γ and δ of S, where $\gamma,\delta\in[0,1]$ with $\gamma<\delta$, (Rehman & Shabir, 2013), if it satisfies the following condition:

$$\max\{\mu(xyz),\gamma\}\geq \min\{\mu(x),\ \mu(y),\ \mu(z),\delta\}$$

for all $x,y,z\in S$.

RESULTS AND DISCUSSION

Implication-based fuzzy ternary subsemigroups: Fuzzy logic is an extension of set theoretic multivalued logic in which the truth values are linguistic variables or terms of the linguistic variable truth. Some operators, for example $\wedge$, $\vee$, $\neg$, $\rightarrow$ in fuzzy logic are also defined by using truth tables and the extension principle can be applied to derive definitions of the operators. In fuzzy logic, the truth value of fuzzy proposition Φ is denoted by $[\Phi]$. For a universe of discourse U, we display the fuzzy logical and corresponding set-theoretical notations used in this paper

$$[x\in\mu]=\mu(x), \quad ...(3)$$

$$[\Phi\wedge\Psi]=\min\{[\Phi],\ [\Psi]\}, \quad ...(4)$$

$$[\Phi\rightarrow\Psi]=\min\{1,\ 1-[\Phi]+[\Psi]\}, \quad ...(5)$$

$$[\forall x\ \Phi(x)]=\inf_{x\in U}[\Phi(x)], \quad ...(6)$$

$$\models\Phi \text{ if and only if } [\Phi]=1 \text{ for all valuations.} \quad ...(7)$$

The truth valuation rules given in equation (5) are those in the Luckasiewicz system of continuous-valued logic. Although, various implication operators have been defined, only a section of them is shown in the following

(a) Gaines-Rescher implication operator (I_{GR}):

$$I_{GR}(a,b)=\begin{cases}1 & \text{if } a\leq b,\\ 0 & \text{otherwise}\end{cases}$$

(b) Gödel implication operator (I_G):

$$I_G(a,b)=\begin{cases}1 & \text{if } a\leq b,\\ b & \text{otherwise}\end{cases}$$

(c) The contraposition of Gödel implication operator (I_{cG}):

$$I_{cG}(a,b)=\begin{cases}1 & \text{if } a\leq b,\\ 1-a & \text{otherwise}\end{cases}$$

(d) The Luckasiewicz implication operator (I_{LI}):

$$I_{LI}(a,b)=\begin{cases}1 & \text{if } a\leq b,\\ 1-a+b & \text{otherwise}\end{cases}$$

for all $a, b \in [0,1]$. Ying (1991) introduced the concept of fuzzifying topology. We can expand the idea to ternary semigroups, and we define fuzzifying ternary subsemigroup as follows:

Definition 2.

A fuzzy set μ in S is called a fuzzifying ternary subsemigroup of S if it satisfies the following condition:

$$\models [x \in \mu] \wedge [y \in \mu] \wedge [x \in \mu] \rightarrow [xyz \in \mu] \qquad ...(8)$$

for all $x, y, z \in S$.

Obviously, condition (8) is equivalent to (1). Therefore a fuzzifying ternary subsemigroup is an ordinary fuzzy ternary subsemigroup. In Ying (1988) the concept of t-tautology is introduced, that is,

$\models_t \Phi$ if and only if $[\Phi] \geq t$ for all valuations.

for all $x, y, z \in S$.

Now we extend the concept of implication-based fuzzy ternary subsemigroups in the following way:

Definition 3.

Let μ be a fuzzy set in S and $t \in (0,1]$. Then μ is called a t-implication-based fuzzy ternary subsemigroup of S if and only if it satisfies

$$\models_t [x \in \mu] \wedge [y \in \mu] \wedge [x \in \mu] \rightarrow [xyz \in \mu] \qquad ...(9)$$

for all $x, y, z \in S$.

Let I be an implication operator. Clearly, μ is a t-implication-based fuzzy ternary subsemigroup of S if and only if it satisfies

$$I(\min\{\mu(x),\ \mu(y),\ \mu(z)\}, \mu(xyz)) \geq t$$

for all $x, y, z \in S$.

Example 2.

Consider the ternary semigroup $S = \{-i, 0, i\}$ under the usual multiplication of complex numbers. Define a fuzzy set μ in S as follows:

$$\mu : S \rightarrow [0,1], x \rightarrow \begin{cases} 0.6 & \text{if } x = -i \\ 0.2 & \text{if } x = 0 \\ 0.7 & \text{if } x = i. \end{cases}$$

Then μ is a t-implication-based fuzzy ternary subsemigroup of S for all $t \in (0, 0.6]$ under the Gödel implication operator I_G. Also μ is a 0.3-implication-based fuzzy ternary subsemigroup of S under the contraposition of Gödel implication operator I_{cG}. We also see that μ is a t-implication-based fuzzy ternary subsemigroup of S for all $t \in (0, 0.9]$ under the Luckasiewicz implication operator I_{LI}.

Example 3.

Consider the ternary semigroup S of Example 2. Define a fuzzy set μ in S as follows:

$$\mu : S \rightarrow [0,1], x \rightarrow \begin{cases} 0.9 & \text{if } x = -i \\ 0.3 & \text{if } x = 0 \\ 0.7 & \text{if } x = i. \end{cases}$$

By routine calculations, we know that μ is a t-implication-based fuzzy ternary subsemigroup of S for all $t \in (0, 0.7]$ under the Gödel implication operator I_G. Also μ is a 0.1-implication-based fuzzy ternary subsemigroup of S under the contraposition of Godel implication operator I_{cG}. We also see that μ is a t-implication-based fuzzy ternary subsemigroup of S for all $t \in (0, 0.8]$ under the Luckasiewicz implication operator I_{LI}.

Note that if $t_1, t_2 \in (0,1]$ with $t_1 > t_2$, then every t_1-implication-based fuzzy ternary subsemigroup of S is a t_2-implication-based fuzzy ternary subsemigroup of S. But the converse is false. In fact, in Example 3, the t-implication-based fuzzy ternary subsemigroup of S for all $t \in (0, 0.7]$ under the Gödel implication operator I_G is not a t-implication-based fuzzy ternary subsemigroup of S for all $t \in (0.7, 1]$ under the Gödel implication operator I_G since

$$I_G(\min\{\mu(-i),\ \mu(-i),\ \mu(-i)\}, \mu((-i)(-i)(-i)))$$

$$= I_G(0.9, 0.7) = 0.7 \ngeq t$$

Lemma 1. (Rehman & Shabir, 2012). A fuzzy set μ in S is an $(\in, \in \vee q)$-fuzzy ternary subsemigroup of S if and only if it satisfies:

$$(\forall x, y, z \in S)\,(\mu(xyz) \geq \min\{\mu(x),\ \mu(y),\ \mu(z),\ 0.5\}) \qquad ...(10)$$

Theorem 1. For any fuzzy set μ in S, if $I = I_G$ and μ is an $(\in, \in \vee q)$-fuzzy ternary subsemigroup of S, then μ is a t-implication-based fuzzy ternary subsemigroup of S for all $t \in (0, 0.5]$.

Proof. Let $t \in (0, 0.5]$ and assume that μ is an $(\in, \in \vee q)$-fuzzy ternary subsemigroup of *S*. Then

$$\mu(xyz) \geq \min\{\mu(x), \ \mu(y), \ \mu(z), \ 0.5\}$$

for all $x, y, z \in S$. If $\min\{\mu(x), \ \mu(y), \ \mu(z)\} \leq 0.5$, then

$$\mu(xyz) \geq \min\{\mu(x), \ \mu(y), \ \mu(z)\}$$

and so

$$I_G(\min\{\mu(x), \ \mu(y), \ \mu(z)\}, \mu(xyz)) = 1 \geq t.$$

Now suppose that $\min\{\mu(x), \ \mu(y), \ \mu(z)\} > 0.5$. Then $\mu(xyz) \geq 0.5$, and either $\mu(xyz) \geq \min\{\mu(x), \mu(y), \ \mu(z)\}$ or $\mu(xyz) < \min\{\mu(x), \ \mu(y), \ \mu(z)\}$. If $\mu(xyz) \geq \min\{\mu(x), \ \mu(y), \ \mu(z)\}$, then

$$I_G(\min\{\mu(x), \ \mu(y), \ \mu(z)\}, \mu(xyz)) = 1 \geq t.$$

If $\mu(xyz) < \min\{\mu(x), \ \mu(y), \ \mu(z)\}$, then

$$I_G(\min\{\mu(x), \ \mu(y), \ \mu(z)\}, \mu(xyz)) = \mu(xyz) \geq 0.5 \geq t.$$

Therefore μ is a *t*-implication-based fuzzy ternary subsemigroup of *S* for all $t \in (0, 0.5]$. ■

Corollary 1. For any fuzzy set μ in *S* if the level set

$$U(\mu; t) := \{x \in S \mid \mu(x) \geq t\}$$

is a ternary subsemigroup of *S*, then μ is a *t*-implication-based fuzzy ternary subsemigroup of *S* for all $t \in (0, 0.5]$ under the Gödel implication operator.

Proof. Straightforward. ■

In the following example it is shown that there exists a fuzzy set μ in *S*, which is a *t*-implication-based fuzzy ternary subsemigroup of *S* for $t \in (0, 0.4]$ under the Gödel implication operator I_G. But μ is not an $(\in, \in \vee q)$-fuzzy ternary subsemigroup of *S*. This shows that the partial converse of Theorem 1 is not true.

Example 4.

Let $S = \{0, a, b, c, 1\}$ and $xyz = (x * y) * z = x * (y * z)$ for all $x, y, z \in S$, where $*$ is defined by the following table:

$*$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	a	a
b	0	0	b	b	b
c	0	0	b	c	c
1	0	a	b	c	1

Then *S* is a ternary semigroup. Define a fuzzy set μ in *S* as follows:

$$\mu : S \to [0,1], \quad x \mapsto \begin{cases} 0.4 & \text{if } x = 0, \\ 0.9 & \text{if } x = a, \\ 0.6 & \text{if } x = b, \\ 0.7 & \text{if } x = c, \\ 0.2 & \text{if } x = 1. \end{cases}$$

Routine calculations show that μ is a *t*-implication-based fuzzy ternary subsemigroup of *S* for all $t \in (0, 0.4]$ under the Gödel implication operator, but μ is not an $(\in, \in \vee q)$-fuzzy ternary subsemigroup of *S*.

Theorem 2. For any fuzzy set μ in *S* and $I = I_G$, if there exists $t \in [0.5, 1]$ such that μ is a *t*-implication-based fuzzy ternary subsemigroup of *S*, then μ is an $(\in, \in \vee q)$-fuzzy ternary subsemigroup of *S*.

Proof. Let $t \in [0.5, 1]$ be such that μ is a *t*-implication-based fuzzy ternary subsemigroup of *S*. Then

$$I_G(\min\{\mu(x), \ \mu(y), \ \mu(z)\}, \mu(xyz)) \geq t$$

for all $x, y, z \in S$, and so either $I_G(\min\{\mu(x), \ \mu(y), \mu(z)\}, \mu(xyz)) = 1$, that is,

$$\mu(xyz) \geq \min\{\mu(x), \ \mu(y), \ \mu(z)\}$$

or $I_G(\min\{\mu(x), \ \mu(y), \ \mu(z)\}, \mu(xyz)) = \mu(xyz) \geq t \geq 0.5$. Hence

$$\mu(xyz) \geq \min\{\mu(x), \ \mu(y), \ \mu(z), 0.5\}.$$

Using Lemma 1, we know that μ is a $(\in, \in \vee q)$-fuzzy ternary subsemigroup of *S*. ■

Corollary 2. For any $t \in [0.5, 1]$, if μ is a *t*-implication-based fuzzy ternary subsemigroup of *S* under the Gödel implication operator I_G, then μ is a fuzzy ternary subsemigroup of *S* with thresholds $\gamma = 0$ and $\delta \in (0, 0.5]$.

Proof. Straightforward. ■

Corollary 3. For any $t \in [0.5, 1]$, if μ is a *t*-implication-based fuzzy ternary subsemigroup of *S* under the Gödel implication operator I_G, then the level set

$$U(\mu; k) := \{x \in S \mid \mu(x) \geq k\}$$

is a ternary subsemigroup of *S* for all $k \in (0, 0.5]$.

Proof. Straightforward. ■

If $t \in [0.5, 1]$, then the converse of Theorem 2 may not be true in general as seen in the following example.

Example 5.

Consider the ternary semigroup *S* of Example 2. Define a fuzzy set μ in *S* as follows:

$$\mu : S \to [0,1], x \to \begin{cases} 0.8 & \text{if } x = -i \\ 0.3 & \text{if } x = 0 \\ 0.5 & \text{if } x = i \end{cases}$$

Routine calculations show that μ is an $(\in, \in \vee q)$-fuzzy ternary subsemigroup of *S*. But μ is not a *t*-implication-based fuzzy ternary subsemigroup of *S* for any $t \in [0.5, 1]$ under the Gödel implication operator I_G, since

$$\begin{aligned} I_G\left(\min\left\{\mu(-i),\ \mu(-i),\ \mu(-i)\right\}, \mu((-i)(-i)(-i))\right) &= I_G(0.8, 0.5) \\ &= 0.5 \ngeq t \end{aligned}$$

for any $t \in [0.5, 1]$.

Combining Theorems 1 and 2 we have the following corollary.

Corollary 4. For any fuzzy set μ in *S*, if $I = I_G$, then μ is a 0.5-implication based fuzzy ternary subsemigroup of *S* if and only if μ is a fuzzy ternary subsemigroup of *S* with thresholds $\gamma = 0$ and $\delta = 0.5$, that is, μ is an $(\in, \in \vee q)$-fuzzy ternary subsemigroup of *S*.

Theorem 3. Consider $I = I_{cG}$ and let $t \in [0.5, 1]$. If μ is a *t*-implication based fuzzy ternary subsemigroup of *S*, then μ is a fuzzy ternary subsemigroup with thresholds $\gamma = t$ and δ, where $\delta = \sup_{x \in S} \mu(x)$.

Proof. Let $t \in [0.5, 1]$ and assume that μ is a *t*-implication-based fuzzy ternary subsemigroup of *S*. Then

$$I_{cG}\left(\min\left\{\mu(x),\ \mu(y),\ \mu(z)\right\}, \mu(xyz)\right) \geq t$$

for all $x, y, z \in S$, and so either $I_{cG}(\min\{\mu(x), \mu(y),\ \mu(z)\}, \mu(xyz)) = 1$, that is,

$$\min\left\{\mu(x),\ \mu(y),\ \mu(z)\right\} \leq \mu(xyz)$$

or $1 - \min\{\mu(x),\ \mu(y),\ \mu(z)\} = I_{cG}(\min\{\mu(x),\ \mu(y)\ \mu(z)\}, \mu(xyz)) \geq t$, that is,

$$\min\left\{\mu(x),\ \mu(y),\ \mu(z)\right\} \leq 1 - t \leq t$$

since $t \in [0.5, 1]$. It follows that

$$\max\left\{\mu(xyz), t\right\} \geq \min\left\{\mu(x),\ \mu(y),\ \mu(z)\right\} = \min\left\{\mu(x), \mu(y),\ \mu(z),\ \delta\right\}.$$

Therefore μ is a fuzzy ternary subsemigroup of *S* with thresholds $\gamma = t$ and $\delta = \sup_{x \in S} \mu(x)$. ■

If $t \in [0.5, 1]$, then the converse of Theorem 3 may not be true in general as seen in the following example.

Example 6.

Consider the ternary semigroup *S* as given in Example 2. Define a fuzzy set μ in *S* as follows:

$$\mu : S \to [0,1], x \to \begin{cases} 0.4 & \text{if } x = -i \\ 0.6 & \text{if } x = 0 \\ 0.3 & \text{if } x = i. \end{cases}$$

Routine calculations show that μ is a fuzzy ternary subsemigroup of *S* with thresholds $\gamma = t$ and $\delta = \sup_{x \in S} \mu(x)$, for all $t \in [0.5, 1]$. But μ is not a *t*-implication based fuzzy ternary subsemigroup of *S* for $t \in (0.6, 1]$. Since

$$\begin{aligned} I_{cG}\left(\min\left\{\mu(-i),\ \mu(-i),\ \mu(-i)\right\}, \mu((-i)x(-i)y(-i))\right) &= I_{cG}(0.4, 0.3) \\ &= 0.6 \ngeq t \end{aligned}$$

for all $t \in (0.6, 1]$.

Now we prove:

Theorem 4. Consider $I = I_{cG}$ and let μ be a fuzzy set in *S*. For every $t \in [0.5, 1]$, if μ is a *t*-implication-based fuzzy ternary subsemigroup of *S*, then μ is a fuzzy ternary subsemigroup with thresholds $\gamma = 1 - t$ and $\delta = \sup_{x \in S} \mu(x)$.

Proof. Assume that μ is a *t*-implication-based fuzzy ternary subsemigroup of *S* for $t \in (0, 0.5]$. Then

$$I_{cG}\left(\min\left\{\mu(x),\ \mu(y),\ \mu(z)\right\}, \mu(xyz)\right) \geq t$$

for all $x, y, z \in S$, which implies that either $\min\{\mu(x),\ \mu(y),\ \mu(z)\} \leq \mu(xyz)$ or

$$1 - \min\left\{\mu(x),\ \mu(y),\ \mu(z)\right\} = I_{cG}\left(\min\left\{\mu(x),\right.\right.$$

$\mu(y),\ \mu(z)\}, \mu(xyz)) \geq t$ and so $\min\{\mu(x),\ \mu(y),\ \mu(z)\} \leq 1 - t$. It follows that

$$\max\left\{\mu(xyz), 1 - t\right\} \geq \min\left\{\mu(x),\ \mu(y),\ \mu(z)\right\} = \min\left\{\mu(x),\ \mu(y),\ \mu(z),\ \delta\right\}.$$

Therefore μ is a fuzzy ternary subsemigroup of *S* with thresholds $\gamma = 1 - t$ and $\delta = \sup_{x \in S} \mu(x)$. ■

Corollary 5. For every $t \in (0, 0.5)$, if μ is a *t*-implication-based fuzzy ternary subsemigroup of S under the contraposition of Gödel implication operator I_{cG}, then μ is a fuzzy ternary subsemigroup with thresholds $\gamma = 1 - t$ and $\delta = 1$.

For the converse of Theorem 3, we have the following theorem.

Theorem 5. Consider $I = I_{cG}$ and let μ be a fuzzy set in S. For every $t \in (0, 0.5)$, if μ is a fuzzy ternary subsemigroup of S with thresholds $\gamma = t$ and $\delta = \sup_{x \in S} \mu(x)$, then μ is a t-implication-based fuzzy ternary subsemigroup of S.

Proof. Let $t \in (0, 0.5]$ and assume that μ is a fuzzy ternary subsemigroup of S with thresholds $\gamma = t$ and $\delta = \sup_{x \in S} \mu(x)$. Then, for all $x, y, z \in S$, we have

$$\max\{\mu(xyz), t\} \geq \min\{\mu(x),\ \mu(y),\ \mu(z), \delta\} = \min\{\mu(x),\ \mu(y),\ \mu(z)\}.$$

If $\mu(xyz) \geq t$, then $\mu(xyz) \geq \min\{\mu(x),\ \mu(y),\ \mu(z)\}$ and so

$$I_{cG}(\min\{\mu(x),\ \mu(y),\ \mu(z)\}, \mu(xyz)) = 1 \geq t.$$

If $\mu(xyz) < t$, then $\min\{\mu(x),\ \mu(y),\ \mu(z)\} \leq t$.

Hence if $\min\{\mu(x),\ \mu(y),\ \mu(z)\} \leq \mu(xyz)$, then

$$I_{cG}(\min\{\mu(x),\ \mu(y),\ \mu(z)\}, \mu(xyz)) = 1 \geq t.$$

If $\min\{\mu(x),\ \mu(y),\ \mu(z)\} > \mu(xyz)$, then

$$I_{cG}(\min\{\mu(x),\ \mu(y),\ \mu(z)\}, \mu(xyz)) = 1 - \min\{\mu(x),\ \mu(y),\ \mu(z)\} \geq 1 - t \geq t.$$

Consequently μ is a t-implication-based fuzzy ternary subsemigroup of S for every $t \in (0, 0.5]$. ■

Corollary 6. For every $t \in (0, 0.5]$, if μ is a fuzzy ternary subsemigroup of S with thresholds $\gamma = t$ and $\delta = 1$, then μ is a t-implication-based fuzzy ternary subsemigroup of S under the contraposition of Gödel implication operator I_{cG}.

Combining Corollaries 5 and 6, we have the following corollary.

Corollary 7. For any fuzzy set μ in S, if $I = I_{cG}$, then μ is a 0.5-implication based fuzzy ternary subsemigroup of S if and only if μ is a fuzzy ternary subsemigroup of S with thresholds $\gamma = t$ and $\delta = 1$.

If $t \in (0, 0.5)$, then the converse of Theorem 5 may not be true in general as seen in the following example.

Example 7.

Consider the ternary semigroup S as given in Example 2. Define a fuzzy set μ in S as follows:

$$\mu : S \to [0,1], x \to \begin{cases} 0.3 & \text{if } x = -i \\ 0.5 & \text{if } x = 0 \\ 0.4 & \text{if } x = i. \end{cases}$$

Routine calculations show that μ is a t-implication-based fuzzy ternary subsemigroup of S for $t \in (0, 0.5)$. But if $t \in (0, 0.4)$, then

$$\max\{\mu(iii), t\} \ngeq \min\{\mu(i), \mu(i), \mu(i)\}$$

Theorem 6. Consider $I = I_{GR}$ and let $t \in (0, 1]$. If μ is a t-implication-based fuzzy ternary subsemigroup of S, then μ is a fuzzy ternary subsemigroup of S.

Proof. Let $t \in (0, 1]$ be such that μ is a t-implication-based fuzzy ternary subsemigroup of S under the Gaines-Rescher implication operator I_{GR}. Then

$$I_{GR}(\min\{\mu(x),\ \mu(y),\ \mu(z)\}, \mu(xyz)) \geq t.$$

Since $t \neq (0, 1]$, it follows that $I_{GR}(\min\{\mu(x),\ \mu(y),\ \mu(z)\}, \mu(xyz)) = 1$ and so that $\mu(xyz) \geq \min\{\mu(x),\ \mu(y),\ \mu(z)\}$. Therefore μ is a fuzzy ternary subsemigroup of S. ■

Corollary 8. For any $t \in (0, 1]$, if μ is a t-implication-based fuzzy ternary subsemigroup of S under the Gaines-Rescher implication operator I_{GR}, then the set

$$U(\mu; t) := \{x \in S \mid \mu(x) \geq t\}$$

is a ternary subsemigroup of S.

Proof. Straightforward. ■

Theorem 7. Every fuzzy ternary subsemigroup of S is a t-implication-based fuzzy ternary subsemigroup for all $t \in (0, 1]$ under the Gaines-Rescher implication operator I_{GR}.

Proof. Straightforward. ■

The following corollary is by Theorems 6 and 7.

Corollary 9. A fuzzy set in S is a 0.5-implication-based fuzzy ternary subsemigroup of S under the Gaines-Rescher implication operator I_{GR} if and only if it is a fuzzy ternary subsemigroup of S.

Theorem 8. Every fuzzy ternary subsemigroup of S is a t-implication-based fuzzy ternary subsemigroup for all $t \in (0,1]$ under the Luckasiewicz implication operator I_{LI}.

Proof. Straightforward. ■

The following example shows that for a fuzzy set μ in S there exists $t \in (0,1]$ such that

(1) μ is an $(\in, \in \vee q)$-fuzzy ternary subsemigroup of S.
(2) μ is not a t-implication-based fuzzy ternary subsemigroup of S under the Luckasiewicz implication operator I_{LI}.

Example 8. Consider the ternary semigroup S of Example 4. Define a fuzzy set μ in S as follows:

$$\mu : S \rightarrow [0,1], \quad x \mapsto \begin{cases} 0.6 & \text{if } x = 0, \\ 0.9 & \text{if } x = a, \\ 0.8 & \text{if } x = b, \\ 0.7 & \text{if } x = c, \\ 0.3 & \text{if } x = 1. \end{cases}$$

Routine calculations show that μ is an $(\in, \in \vee q)$- fuzzy ternary subsemigroup of S. But μ is not a 0.91-implication-based fuzzy ternary subsemigroup of S under the Luckasiewicz implication operator I_{LI} since

$$\begin{aligned} I_{LI}\left(\min\left\{\mu\left(a\right),\ \mu\left(b\right),\ \mu\left(c\right)\right\},\mu\left(abc\right)\right) &= I_{LI}\left(0.7, 0.6\right) \\ &= 1 - 0.7 + 0.6 \\ &= 0.9 \ngeq 0.91. \end{aligned}$$

We provide conditions for an $(\in, \in \vee q)$-fuzzy ternary subsemigroup of S to be a t-implication-based fuzzy ternary subsemigroup of S under the Luckasiewicz implication operator I_{LI}.

Theorem 9. Let μ be an $(\in, \in \vee q)$-fuzzy ternary subsemigroup of S. If there exist $x, y, z \in S$ such that $\min\left\{\mu\left(x\right),\ \mu\left(y\right),\ \mu\left(z\right)\right\} > \mu\left(xyz\right)$, and let $\omega = 1 - \min\left\{\mu\left(x\right),\ \mu\left(y\right),\ \mu\left(z\right)\right\} + \mu\left(xyz\right)$. Then μ is an t-implication-based fuzzy ternary subsemigroup of S for all $t \in (0, \omega]$.

Proof. Assume that μ is an $(\in, \in \vee q)$-fuzzy ternary subsemigroup of S. Then $\mu\left(xyz\right) \geq \min\left\{\mu\left(x\right),\ \mu\left(y\right),\ \mu\left(z\right),\ 0.5\right\} = \min\left\{\min\left\{\mu\left(x\right),\ \mu\left(y\right),\ \mu\left(z\right)\right\},\ 0.5\right\}$, for all $x, y, z \in S$. Suppose that $\min\left\{\mu\left(x\right),\ \mu\left(y\right),\ \mu\left(z\right)\right\} \leq 0.5$. Then $\mu\left(xyz\right) \geq \min\left\{\mu\left(x\right),\ \mu\left(y\right),\ \mu\left(z\right)\right\}$ and so

$$I_{LI}\left(\min\left\{\mu\left(x\right),\ \mu\left(y\right),\ \mu\left(z\right)\right\},\mu\left(xyz\right)\right) = 1 \geq t$$

for all $t \in (0, \omega]$. Assume that $\min\left\{\mu\left(x\right),\ \mu\left(y\right),\ \mu\left(z\right)\right\} > 0.5$ for all $x, y, z \in S$. Then $\mu\left(xyz\right) \geq 0.5$.

Thus we have two cases:

(1) $\mu\left(xyz\right) \geq \min\left\{\mu\left(x\right),\ \mu\left(y\right),\ \mu\left(z\right)\right\}$,
(2) $\mu\left(xyz\right) < \min\left\{\mu\left(x\right),\ \mu\left(y\right),\ \mu\left(z\right)\right\}$.

First case implies that

$$I_{LI}\left(\min\left\{\mu\left(x\right),\ \mu\left(y\right),\ \mu\left(z\right)\right\},\mu\left(xyz\right)\right) = 1 \geq t$$

for all $t \in (0, \omega]$. The second case induces

$$\begin{aligned} &I_{LI}\left(\min\left\{\mu\left(x\right),\ \mu\left(y\right),\ \mu\left(z\right)\right\},\mu\left(xyz\right)\right) \\ &\quad = 1 - \min\left\{\mu\left(x\right),\ \mu\left(y\right),\ \mu\left(z\right)\right\} + \mu\left(xyz\right) \\ &\quad = \omega \geq t \end{aligned}$$

for all $t \in (0, \omega]$. Therefore μ is a t-implication-based fuzzy ternary subsemigroup of S for all $t \in (0, \omega]$ under the Luckasiewicz implication operator I_{LI}.

Remark 1. The presented work has applications in the following disciplines:

(i) Computer science and information theory
(ii) Artificial intelligence
(iii) Temperature control system

CONCLUSION

Ternary and n-ary generalisations of algebraic structures are the most natural ways for further development and deeper understanding of their fundamental properties. In the current study, by using four implication operators, that is, Gaines-Rescher implication operator, Gödel implication operator, the contraposition of Gödel implication operator and the Luckasiewicz implication operator, the implication-based fuzzy ternary subsemigroups are considered. We also describe connections between these types of implicative operators and fuzzy [$(\in, \in \vee q)$- fuzzy] ternary subsemigroups. Conditions for a fuzzy ternary subsemigroup with thresholds 0 and 0.5 to be an implication-based fuzzy ternary subsemigroups under the Luckasiewicz implication operator are provided, and thus we answered the question posed in the Introduction section.

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