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SUPPORT POINTS OF SUBORDINATION FAMILIES

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Abstract

A general theorem for support points of subordination families is proved. This theorem provides a unified approach to the support points of the families subordinate to starlike functions, close-to-convex functions of order $\beta(\geq 1)$, and functions having boundary rotation at most $k\pi (k > 2)$. The results for starlike functions and close-to-convex functions were previously known by different methods.

1 INTRODUCTION

Let $\mathcal{A}$ denote the set of functions analytic in $\mathcal{D} = \{z \in \mathbb{C} : |z| < 1\}$. Then $\mathcal{A}$ is a locally convex linear topological space under the topology given by uniform convergence on compact subsets of $\mathcal{D}$.

Let

$$\mathcal{B}_o = \{ \phi \in \mathcal{A} : \phi(0) = 0, |\phi(z)| < 1 \text{ for } z \text{ in } \mathcal{D}\}.$$

A function f in $\mathcal{A}$ is said to be subordinate to a function g in $\mathcal{A}$ if there is a function ϕ in $\mathcal{B}_o$ such that $f = go\phi$. This is denoted by $f \prec g$. Let

$$s(g) = \{ f \in \mathcal{A} : f \prec g\}.$$

Let $\mathcal{F}$ be a compact subset of $\mathcal{A}$. Let

$$s(\mathcal{F}) = \{ f \in \mathcal{A} : f \prec g \text{ for some } g \text{ in } \mathcal{F} \}.$$

A function f in $\mathcal{F}$ is a support point of $\mathcal{F}$ if there is a continuous linear functional J on $\mathcal{A}$ such that

$$Re\ J(f) = max\ \{Re\ J(g) : g \in \mathcal{F} \}$$

and $Re\ J$ is nonconstant on $\mathcal{F}$. We use $\Sigma\ \mathcal{F}$ and $\Sigma\ \{\mathcal{F}, J\}$ to denote the set of support points of $\mathcal{F}$ and the support points of $\mathcal{F}$ associated with a specific continuous linear functional J, respectively. We also use $\overline{co}\ \mathcal{F}$ to denote the closed convex hull of $\mathcal{F}$. Let $U = \{ z \in \mathcal{C} : | z | = 1 \}$. Following [6], we let

$$\mathcal{R} = \{ f \in \mathcal{A} : \overline{co}\ s(f) = \{\textstyle\int_U f(xz)d\mu(x) : \mu \text{ is a probability measure on } U\} \}$$

and

$$\mathcal{R}_\Sigma = \{f \in \mathcal{A} : \Sigma\ s(f) = \{ f(xz) : | x | = 1\}\}.$$

Let S, St, $C(\beta)$, and ϑ_k denote the normalized families of univalent functions, starlike function, close-to-convex functions of order β, and functions having boundary rotation at most $k\pi$, respectively.

It was noted in [1] that, if $\mathcal{F}$ is a compact subset of $\mathcal{A}$ such that $f(0) = 0$ and $f'(0) = 1$ whenever $f \in \mathcal{F}$ and if J is a continuous linear functional of the form $J(f) = af(0) + bf'(0)$ whenever $f \in \mathcal{A}$ $(a, b \in \mathcal{C}, b \neq 0)$, then there is an unique x with $| x | = 1$ such that for any $F \in \mathcal{F}, F(xz)$ is a support point of $s(\mathcal{F})$ associated with J.

Now let J be a continuous linear functional not of the form $J(f) = af(0) + bf'(0)$ $(a, b \in \mathcal{C})$. In [4], Hallenbeck and MacGregor proved that

$$\sum \{s(St), J\} \subseteq \left\{\frac{xz}{(1 - yz)^2} : | x | = | y | = 1\right\}$$

(This result can also be found in [5, p.113].) In [1], Abu-Muhana and Hallenbeck proved that, if $\mathcal{F}$ is a compact subset of $\mathcal{A}$ such that $St \subseteq \mathcal{F} \subseteq S$, then

$$\sum \{s(\mathcal{F}),\ J\} \subseteq \{F o \phi : F \in \Sigma\ \mathcal{F},\ \phi \in \Sigma\ \mathcal{B}_0 \}.$$

Then they used this result to prove that

$$\sum \{s(C(1)),\ J\} \subseteq \left\{\omega\ \frac{z - \frac{1}{2}(x + y)z^2}{(1 - yz)^2} : | x | = | y | = | \omega | = 1,\ x \neq y\right\}.$$

It was proved in [8, p.56], that for $\beta > 1$

$$\sum \{s(C(\beta)), J\} \subseteq \left\{\frac{\omega}{(\beta + 1)(y - x)}\left[\left(\frac{1 - xz}{1 - yz}\right)^{\beta+1} - 1\right] : | x | = | y | = | \omega | = 1, x \neq y\right\}.$$

Note that, for $\beta > 1$, C(β) is no longer contained in $\mathcal{S}$.

In this note we prove a general theorem which covers all of the results mentioned above. A particularly interesting case is the assertion that, for $k \geq 2$,

$$\sum\{s(\vartheta_k), J\} \subseteq \{F(xz) : F \in \Sigma\, \vartheta_k, |x| = 1\}.$$

2 MAIN RESULT

We shall need the following result proved in [6].

Theorem A. Let $F \in \mathcal{R}$, $F(z) = \Sigma_{n=0}^{\infty} A_n z^n$. If $A_N \neq 0$ where $N \geq 1$, then $A_M \neq 0$ for every $M > N$.

All of the families mentioned in the introduction and many other well-known sub families of A satisfy the following conditions:

(1) $f(0) = 0$ and $f'(0) = 1$ whenever $f \in \mathcal{F}$

(2) $z \in \mathcal{F}$

(3) $f_x \in \mathcal{F}$ whenever $f \in \mathcal{F}$, where $f_x(z) = \overline{x}\, f(xz)$

Theorem 1.

Let $\mathcal{F}$ be a compact subset of $\mathcal{A}$ satisfying the conditions 1) - 3) such that $\Sigma\, \mathcal{F} \subseteq \mathcal{R}_\Sigma$. If f_0 is a support point of $s(\mathcal{F})$ associated with a continuous linear functional J not of the form $J(f) = a\, f(0) + bf'(0)$ $(a,\ b \in \mathbb{C})$, then $f_0(z) = F_0(x_0 z)$, where $F_0 \in \Sigma\, \mathcal{F}$ and $|x_0| = 1$.

Proof Let $f_0(z) = F_0(\phi_0(z))$ where $F_0 \in \mathcal{F}$ and $\phi_0 \in \mathcal{B}_0$. Since f_0 is a support point of $s(\mathcal{F})$ associated with J, we have $Re\ J\,(F_0(\phi_0)) \geq Re\ J\,(F(\phi_0))$ for all F in $\mathcal{F}$ and $Re\ J$ is nonconstant on $s(\mathcal{F})$.

Let $L(f) = J(f(\phi_0))$ for f in $\mathcal{A}$. Then L is a continuous linear functional on $\mathcal{A}$ and $Re\ L(F_0) \geq Re\ L\,(F)$ for all F in $\mathcal{F}$. Hence, either $F_0 \in \Sigma\, \mathcal{F}$ or $Re\ L$ is constant on $\mathcal{F}$. If $Re\ L$ is constant on $\mathcal{F}$, then $Re\ L(F_0) = Re\ L(\mathcal{F})$ for all F in $\mathcal{F}$, i.e. $Re\ J(F_0(\phi_0)) = Re\ J(F(\phi_0))$ for all F in $\mathcal{F}$. Hence $F(\phi_0)$ is a support point of $s(\mathcal{F})$ associated with J for all F in $\mathcal{F}$. In particular $F_1(\phi_0)$ is a support point of $s(\mathcal{F})$ associated with J where $F_1 \in \Sigma\, \mathcal{F}$. This and the fact $s(F_1) \subseteq s(\mathcal{F})$ yield $Re\ J(F_1(\phi_0)) \geq Re\ J(F_1(\phi))$ for all ϕ in $\mathcal{B}_0$, i. e. $Re\ J$ peaks over $s(F_1)$ at $F_1(\phi_0)$. We claim that $Re\ J$ is nonconstant on $s\,(F_1)$. For if $Re\ J$ is constant on $s(F_1)$, then $Re J(F_1(xz))$ is constant for all $|x| = 1$.

Let $F_1(z) = z + \sum_{n=1}^{\infty} a_n z^n$ (condition 1) and $G(x) = J(F_1(xz))$. Since

$$G(x) = xJ(z) + \sum_{n=1}^{\infty} a_n x^n J(z^n)$$

is analytic in the closed disk $|x| \leq 1$ and $Re\, G(x)$ is constant for $|x| = 1$, we have

$$J(z)x + \sum_{n=2}^{\infty} a_n J(z^n) x^n = 0.$$

This gives $J(z^n) = 0$ for $n = 1, 2, \ldots.$ provided $a_n \neq 0$ for $n = 2, 3, \ldots$.

Since $F_1 \in \mathcal{R}_\Sigma$ implies that $F_1 \in \mathcal{R}$ [6], it follows from Theorem A that $a_n \neq 0$ for $n = 2, 3, \ldots$. Hence $J(z^n) = 0$ for $n = 1, 2, \ldots$ and $Re\, J$ is constant on $s(\mathcal{F})$. This contradiction yields $F_1(\phi_0) \in \Sigma\, s(F_1)$. But $F_1 \in \mathcal{R}_\Sigma$ then gives $F_1(\phi_0) = F_1(x_0 z)$ for some $|x_0| = 1$. Then $F_1'(\phi_0(z)).\, \phi_0'(z) = F_1'(x_0 z).\, x_0$ and so $\phi_0'(0) = x_0$. This implies that $\phi_0(z) = x_0 z$. Hence, if $Re\, L$ is constant on $\mathcal{F}$ then $F(x_0 z)$ is a support point of $s(\mathcal{F})$ associated with J for all F in $\mathcal{F}$. Let $G_x(z) = F_1(xz)$, $|x| = 1$.

Then $G_x \in \mathcal{F}$ (condition 3). This fact together with $z \in \mathcal{F}$ (condition 2) implies that

$$Re\, J(x_0 z) = Re\, J(G_x(x_0 z)), \; for\ all\ |x| = 1.$$

This is equivalent to

$$Re \sum_{n=2}^{\infty} a_n x_0^n J(z^n) x^{n-1} = 0 \; for\ all\ |x| = 1.$$

As before we get $J(z^n) = 0$ for $n = 2, 3, \ldots$ and then J has the form $J(f) = a\, f(0) + b\, f'(0)$. This is a contradiction. Hence $F_0 \in \Sigma\, \mathcal{F}$.

It remains to show that $\phi_0(z) = x_0 z, |x_0| = 1$. We now consider $s(F_0)$. $Re\, J$ peaks over $s(F_0)$ at $F_0(\phi_0)$. If $Re\, J$ is constant on $s(F_0)$ then

$$Re\, J(F_0(xz)) \text{ is constant for all } |x| = 1.$$

Let $F_0(z) = z + \sum_{n=2}^{\infty} c_n z^n$. *Since* $F_0 \in \Sigma\, \mathcal{F}\ (\subseteq \mathcal{R}_\Sigma)$, Theorem A implies that $c_n \neq 0$ for $n = 2, 3, \ldots$. Now

$$Re\ J(F_0(xz)) = Re\{x\ J(z)\} + \sum_{n=2}^{\infty} c_n x^n\ J(z^n)$$

is constant for all $|x| = 1$ implies that $J(z^n) = 0$, $n = 1, 2, \ldots$ .and J is constant on $s(\mathcal{F})$. This is impossible. Hence $F_0(\phi_0) \in \Sigma\ s(F_0)$. Since $F_0 \in \mathcal{R}_\Sigma$ it follows, as before, that $\phi_0(z) = x_0 z$ for some $|x_0| = 1$. This completes the proof.

3 APPLICATIONS

When applied to St, the family of normalized starlike functions, Theorem 1 gives the following result obtained by T. H. MacGregor and D. J. Hallenbeck [5, p.113].

Corollary 1. ([4], [5, p.113]). Let J be a continuous linear functional on $\mathcal{A}$ not of the form $J(f) = af(0) + bf'(0)\ (a, b \in \mathbb{C})$. Then

$$\sum \{ s(St), J\} \subseteq \left\{ \frac{xz}{(1 - yz)^2} : |x| = |y| = 1 \right\}.$$

Proof The conditions 1) - 3) of Theorem 1 are clearly satisfied. It is known that $\Sigma\ (St) = \left\{ \frac{z}{(1-xz)^2} : |x| = 1 \right\}$ [2]. A result in [5, p. 104] implies that $\frac{z}{(1-xz)^2} \in \mathcal{R}_\Sigma$. Now the corollary follows from Theorem 1.

We now apply Theorem 1 to $C(\beta)$, the family of close - to - convex functions of order $\beta (\geq 1)$. Recall that $C(\beta)$ consists of function $f \in \mathcal{A}$ such that $f(0) = 0$, $f'(0) = 1$ and $\left| arg\ \frac{zf'(z)}{e^{i\gamma} s(z)} \right| \leq \beta . \frac{\pi}{2}$ for z in D, for some s in St and some real number γ.

Corollary 2. ([1], [8, p.56]). Let J be a continuous linear functional not of the form $J(f) = a\ f(0) + bf'(0)\ (a, b \in \mathbb{C})$. If $\beta \geq 1$, then

$$\sum \{s(C(\beta)), J\} \subseteq \left\{ \frac{w}{(\beta + 1)(y - x)} \left[\left(\frac{1 - xz}{1 - yz} \right)^{\beta+1} - 1 \right] : |x| = |y| = |w| = 1, x \neq y \right\}.$$

Proof: It is known that ([3], [8, p.56]), for $\beta \geq 1$

$$\sum C(\beta) \subseteq \left\{ \frac{1}{(\beta + 1)(y - x)} \left[\left(\frac{1 - xz}{1 - yz} \right)^{\beta+1} - 1 \right] : |x| = |y| = 1,\ x \neq y \right\}.$$

Hence a result in [5, p.104], and the fact that if $f \in \mathcal{R}_{\Sigma}$, then $af(xz) + b \in \mathcal{R}_{\Sigma}$ $(a, b \in \mathbb{C},\ |x| = 1,\ a \neq 0)$ implies that $\Sigma\, \mathcal{C}\,(\beta) \subseteq \mathcal{R}_{\Sigma}$. Now the corollary follows from Theorem 1.

Let us recall that a function $f \in \mathcal{A}$ belongs to ϑ_k, the family of functions having boundary rotation at most $k\pi$ $(k \geq 2)$, if and only if $f(0) = 0$ and

$$f'(z) = exp \int_{U} -log(1 - xz)\, d\mu(x),$$

where μ is a real measure on U such that $\int_U |d\mu| \leq k$ and $\int_U d\mu = 2$. Note that $\vartheta_2 = \mathcal{K}$, the family of normalized convex function. In the proof of Corollary 3 we use the following theorem proved in [6].

Theorem B Let $F \in \mathcal{R}$ be such that

(1) F is univalent, and

(2) $\mathbb{C} \backslash F\,(\mathcal{D})$ is convex and not a half-plane.

Then $F \in \mathcal{R}_{\Sigma}$.

Corollary 3. Let J be a continuous linear functional not of the form $J(f) = af(0) + bf'(0)\ (a, b \in \mathbb{C})$. If $k > 2$,

then

$$\sum \{\, s(\vartheta_k),\ J \,\} \subseteq \{F(xz) : F \in \Sigma\, \vartheta_k,\ |x| = 1\}.$$

Proof : We shall show that, if $k > 2$, then $\Sigma\, \vartheta_k \subseteq \mathcal{R}_{\Sigma}$.

If $k \geq 4$, then

$$\sum \vartheta_k \subseteq \left\{ \frac{z}{k(y - x)} \left[\left(\frac{1 - xz}{1 - yz} \right)^{\frac{k}{2}} - 1 \right] : |x| = |y| = 1,\ x \neq y \right\} \quad [1].$$

As in Corollary 2, $\Sigma\, \vartheta_k \subseteq \mathcal{R}_{\Sigma}$. Now suppose that $2 < k < 4$. It was shown in [7] that if f is a support point of ϑ_k, then

$$f'(z) = \frac{\prod_{j=1}^{N} (1 - x_j z)^{\gamma_j}}{(1 - y_0 z)^{\frac{k}{2}+1}}$$

Where N is a positive integer, $\sum_{j=1}^{N} \gamma_j = \frac{k}{2} - 1$, $\gamma_j > 0$, $| x_j | = | y_0 | = 1$ for $j = 1, 2, \dots, N$ and $y_0 \notin \{x_1, x_2, \dots, x_n\}$.

It was noted there that these support points have complement convex range. Since $k > 2$, none of these are half - plane mappings. It is well - known that if $2 < k < 4$, then ϑ_k consists of univalent functions. Hence for $2 < k < 4$ Theorem B implies that $\Sigma\, \vartheta_k \subseteq \mathcal{R}_\Sigma$.

So if $k > 2$, then $\Sigma\, \vartheta_k \subseteq \mathcal{R}_\Sigma$. Now Theorem 1 gives the corollary.

Remark. The condition $\Sigma \mathcal{F} \subseteq \mathcal{R}_\Sigma$ of Theorem 1 cannot be replaced by $\Sigma \mathcal{F} \subseteq \mathcal{R}$. This may be seen by letting $\mathcal{F} = \mathcal{K}$. It is known [2] that

$$\sum \mathcal{K} = \left\{ \frac{z}{1 - xz} : | x | = 1 \right\}.$$

A trivial modification of the Herglotz representation formula implies that $\Sigma\mathcal{K} \subseteq \mathcal{R}$. It is easy to see from Theorem 7.3 in [5] that $\frac{z}{1-xz} \notin \mathcal{R}_\Sigma$ for $| x | = 1$. Hence $\Sigma\, \mathcal{K} \subseteq \mathcal{R} \setminus \mathcal{R}_\Sigma$. Theorem 7.20 in [5] states that,

$$\sum s(\mathcal{K}) = \{ho\phi \;:\; h \in \mathcal{K},\ \phi \textit{ is a finite Blaschke product and } \phi(0) = 0\}.$$

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