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RESEARCH ARTICLE

Closed form wave solutions to the time fractional Boussinesq-type equation and the Zakharov-Kuznetsov equation

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Abstract: In this article, we have considered the time fractional Boussinesq-type equation and the (3, 3,.3) time fractional Zakharov-Kuznetsov (FZK) equation, as the model of fluid flow and heat transfer, temperature changes from one place to another, waves on a free-moving fluid surface, periodic and shallow water waves, conservation of mass, ion-acoustic waves in plasma, electromagnetic waves, traffic flow and signal processing waves of optical fibers, etc. Applying the generalised (G'/G)-expansion method and by means of fractional complex transformation, we have examined fresh, useful and further generalised exact travelling wave solutions to the above mentioned equations. We have shown that the method used in this article is a more generalised, straightforward and efficient technique that can be used to establish a large number of fresh solutions of the fractional nonlinear differential equations involved in mathematical physics. We have also discussed the physical explanation for its definite values of emerging parameters of the solutions, which are examined from these equations. Finally we have depicted the 3D and 2D figures to the sense of physical phenomena of the obtained solutions.

Keywords: Fractional nonlinear evolution equations, generalised (G'/G)-expansion method, time fractional Boussinesq type equation, time fractional ZK equation, travelling wave transformation.

INTRODUCTION

Fractional differential equations have gained much importance and opportunity to the researchers in different branches of science and engineering. Today, obtaining exact travelling wave solutions of the fractional nonlinear differential equations is an important task. In order to understand the complex behaviour of fractional differential equations, the exact solutions play a vital role. The fractional differential equations have been broadly used in different fields such as traffic flow, aerodynamics, hydrology, optical fibers, material science, electromagnetics, viscous and elasticity, electrochemistry, physics, signal processing, control theory, fractal dynamics, medicine, chemical kinematics, biology and population, etc. (Resat *et al.*, 2009; Lu, 2012; Alzaidy, 2013; Alam & Akbar, 2014a; 2014b; Firmansyah, 2015; Kaplan *et al.*, 2015).

There are different types of definitions for fractional derivative but in this article we have used Caputo (Caputo & Fabrizio, 2015) derivative and Jumarie's modified Riemann-Liouville (Jumarie, 2006) derivative which are widely used to model many practical problems.

Understanding the complex physical phenomena of exact solutions of fractional nonlinear differential equations is important. Therefore, some well-known useful methods have been enhanced for determining the exact travelling wave solutions to the fractional evolution equations such as, the exp-function method (Zheng, 2013; El-Borai *et al.*, 2015), Adomian decomposition method (El-Sayed *et al.*, 2010), (G'/G)-expansion method (Bekir & Guner, 2013; Alam & Akbar, 2014a; 2014b; 2014c; Youis & Zafar, 2014), fractional sub-equation method (Alzaidy, 2013), modified simple equation method (MSE)

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(Kaplan *et al.*, 2015), first integral method (Lu, 2012; Younis, 2013), Jacobi elliptic method (Zheng & Feng, 2014), differential transformation method (Momani *et al.*, 2007; Rabtah *et al.*, 2010), finite element method (Deng, 2009), variational iteration method (Neamaty *et al.*, 2015), and modified Kudryashov method (Ege & Misirli, 2014), etc.

Recently, to obtain the solutions of the (3, 3, 3), time fractional ZK (Kuznetsov & Petviashvilli, 1970; Zakharov & Kuznetsov,1974) equation was investigated through powerful and efficient techniques such as, the He's homotopy perturbation method (Yildirim & Gulkanat, 2010), reductive perturbation method (Munro & Parkes, 1999), variational iteration method (VIM) (Munro & Parkes, 2000; Batiha, 2009; Molliq *et al.*, 2009), first integral method (Hossein *et al.*, 2015), tanh-coth method (Hossam & Ghany, 2013) and fractional sub-equation method (Ray & Sahoo, 2015), etc. Further, the time fractional Boussinesq-type equation was investigated through the Kudryashov method (Hosseini & Ansari, 2017), tanh-function method (Jaradat *et al.*, 2017), and (G'/G) - expansion method (Alam *et al.*, 2013; 2014; Bekir & Guner, 2013), etc. So far, the fractional Boussinesq-type equation and the (3, 3, 3) time fractional ZK equation have not been investigated by means of the generalised-expansion method.

The objective of this article was to establish the general and new exact travelling wave solutions to the time fractional Boussinesq-type equation and the FZK (3, 3, 3) equation by applying the generalised (G'/G) - expansion method and discuss the physical explanation for the definite values including parameters of the solutions which are determined from the equations. This method is a recently developed, more general and simple method of the fractional nonlinear differential equations involved in mathematical physics to examine the exact travelling wave solutions. The results of this method are simple, straightforward, more general and useful.

METHODOLOGY

Let us consider a general fractional nonlinear differential equation in the form:

$$N\left(w, D_t^{\alpha}w, D_x^{\alpha}w,\ D_y^{\alpha}w, D_t^{2\alpha}w, \dots\right) = 0, \qquad \dots(1)$$

where $w = w(x,y,t)$ is the wave function, N is a polynomial of $w = w(x,y,t)$ and its partial derivatives, which consist of the higher order derivatives and, which generates nonlinear terms of *w* with respect to those, and the subscripts denote variables, which compute partial derivatives. To obtain the solution of equation (1) by using the generalised (G'/G)-expansion method, we have to implement the subsequent steps:

Step 1: Let us consider, $w(x,y,t) = \phi(\xi)e^{i\eta}$ and the travelling wave variable,

$$\xi = k\left(x + y - \frac{ct^{\alpha}}{\Gamma(1+\alpha)}\right) \text{ and } \eta = k\,x + ny - \frac{m\,t^{\alpha}}{\Gamma(1+\alpha)}, \qquad \dots(2)$$

allows us to transform equation (1) into the following ordinary differential equation (ODE):

$$L(w, w', w'', \dots) = 0, \qquad \dots(3)$$

where L is a polynomial of $w(\xi)$ and its derivatives, wherein $w'(\xi) = \frac{dw}{d\xi}$.

Step 2: According to possibility, equation (3) can be integrated term by term one or several times, which yields constants in the right hand side. Those constants can be chosen as zero since we are looking for soliton solutions.

Step 3: Assume that the travelling wave solution of equation (3) can be written in the form:

$$w(\xi) = \sum_{i=0}^{N} a_i (d+H)^i + \sum_{i=1}^{N} b_i (d+H)^{-i}, \qquad \dots(4)$$

where either a_N or b_N may be zero, but both a_N and b_N cannot be zero at a time, $a_i, (i = 0, 1, 2 \dots N)$ and $b_i, (i = 1, 2, \dots N)$ and d are arbitrary constants to be evaluated and $H(\xi)$ is given by

$$H(\xi) = (G'/G) \qquad \dots(5)$$

where $G = G(\xi)$ satisfies the following auxiliary nonlinear ordinary differential equation:

$$PGG'' - QGG' - EG^2 - R(G')^2 = 0 \qquad \dots(6)$$

where the prime stands for derivative with respect to ξ; P, Q, R and E are real parameters.

Step 4: The positive integer *N* that arises in equation (4) can be determined by homogeneous balance between the highest order nonlinear terms and the derivatives of highest order occur in equation (3).

Step 5: Inserting equations (4), (5) and (6) into equation (3) with the value of *N* obtained in

Step 4, we attain polynomial in $(d+H)^N$ $(N=0,1,2,\ldots)$ and $(d+H)^{-N}$ $(N=1,2,3,\ldots)$. Then, we collect each coefficient of the resulted polynomials and equate to zero, which gives a system of algebraic equations for $a_i,(i=0,1,2,\ldots)$, $b_i(i=1,2,3\ldots)$, d and k.

Step 6: Suppose that the value of the constants $a_i,(i=0,1,2,\ldots)$, $b_i(i=1,2,3\ldots)$, d and k can be found by solving the algebraic equations obtained in Step 5. Since the general solution of equation (6) is well known to us, putting the values of $a_i,(i=0,1,2,\ldots)$, $b_i(i=1,2,3\ldots)$, d and k into equation (4), we attain a more general type and new exact travelling wave solutions of the nonlinear fractional differential equation (1).

Using the general solution of equation (6), we attain the following solutions of equation (5):

Set 1: When $Q\neq 0, \varphi=P-R$ and $\Omega=Q^2+4E(P-R)>0$,

$$H(\xi)=(G'/G)=\frac{Q}{2\varphi}+\frac{\sqrt{\Omega}}{2\varphi}\times\frac{C_1 sinh\left(\frac{\sqrt{\Omega}}{2P}\xi\right)+C_2 cosh\left(\frac{\sqrt{\Omega}}{2P}\xi\right)}{C_1 cosh\left(\frac{\sqrt{\Omega}}{2P}\xi\right)+C_2 sinh\left(\frac{\sqrt{\Omega}}{2P}\xi\right)} \quad \ldots(7)$$

Set 2: When $Q\neq 0, \varphi=P-R$ and $\Omega=Q^2+4E(P-R)<0$,

$$H(\xi)=(G'/G)=\frac{Q}{2\varphi}+\frac{\sqrt{-\Omega}}{2\varphi}\times\frac{-C_1 sin\left(\frac{\sqrt{-\Omega}}{2P}\xi\right)+C_2 cos\left(\frac{\sqrt{-\Omega}}{2P}\xi\right)}{C_1 cos\left(\frac{\sqrt{-\Omega}}{2P}\xi\right)+C_2 sin\left(\frac{\sqrt{-\Omega}}{2P}\xi\right)} \quad \ldots(8)$$

Set 3: When $Q\neq 0, \varphi=P-R$ and $\Omega=Q^2+4E(P-R)=0$,

$$H(\xi)=(G'/G)=\frac{Q}{2\varphi}+\frac{C_2}{C_1+C_2\xi} \quad \ldots(9)$$

Set 4: When $Q=0, \varphi=P-R$ and $\Delta=\varphi E>0$,

$$H(\xi)=(G'/G)=\frac{\sqrt{\Delta}}{\varphi}\times\frac{C_1 sinh\left(\frac{\sqrt{\Delta}}{P}\xi\right)+C_2 cosh\left(\frac{\sqrt{\Delta}}{P}\xi\right)}{C_1 cosh\left(\frac{\sqrt{\Delta}}{P}\xi\right)+C_2 sinh\left(\frac{\sqrt{\Delta}}{P}\xi\right)} \quad \ldots(10)$$

Set 5: When $Q=0, \varphi=P-R$ and $\Delta=\varphi E<0$,

$$H(\xi)=(G'/G)=\frac{\sqrt{-\Delta}}{\varphi}\times\frac{-C_1 sin\left(\frac{\sqrt{-\Delta}}{P}\xi\right)+C_2 cos\left(\frac{\sqrt{-\Delta}}{P}\xi\right)}{C_1 cos\left(\frac{\sqrt{-\Delta}}{P}\xi\right)+C_2 sin\left(\frac{\sqrt{-\Delta}}{P}\xi\right)} \quad \ldots(11)$$

RESULTS AND DISCUSSION

In this section, we have examined the new generalised solutions for the time-fractional Boussinesq-type equation and the (3, 3, 3) time fractional Zakharov-Kuznetsov equations and their usefulness.

The fractional Boussinesq-type equation

In this sub-section, we have examined some new close form wave solutions to the fractional Boussinesq-type equation by means of the generalised (G'/G)-expansion method. Let us suppose the time fractional Boussinesq-type equation of the form:

$$D_t^{2\alpha}u-(u)_{xx}-(u^2)_{xx}+[u(u)_{xx}]_{xx}=0;\ t>0, 0<\alpha\leq 1, \quad \ldots(12)$$

where α represents fractional order type derivative. This equation is an important model to understand the dynamics of fluid flow and heat transfer. The behaviour of the model depends on temperature variation and nature of different types of waves such as, waves on free-moving fluid surface, periodic and shallow water waves, signal processing waves through optical fibers, conservation of mass, and acceleration due to gravity, etc.

Applying the traveling transformation [equation (2)], the equation (12) is converted into the following nonlinear ODE:

$$(m^2-k^2)u''-k^2(u^2)''+k^4(u\,u'')''=0, \quad \ldots(13)$$

The equation (13) implies that

$$(m^2-k^2)u-k^2u^2+k^4uu''+c=0, \quad \ldots(14)$$

where c is an integrating constant.

Now, balancing the highest order linear and nonlinear terms occurring in equation (14), yields N = –2. Therefore, we will make use of a new transformation $u=v^{-2}$, which converts the equation (14) into the following nonlinear ODE:

$$(m^2-k^2)v^4-k^2v^2+6k^4(v')^2-2k^4v\,v''+C=0, \quad \ldots(15)$$

Now, balancing between the highest order nonlinear terms occurring in equation (15), yields $N=-2$. Then, the solution of equation (15) turns into the form:

$$v=b_1(d+H)^{-1}+a_0+a_1(d+H), \quad \ldots(16)$$

where a_0, a_1 and b_1 are arbitrary constants to be determined, such that either a_1 or b_1 may be zero, but both a_1 and b_1 cannot be zero at a time.

Substituting equation (16) together with equations (5) and (6) into equation (15), the left hand side is transformed into polynomials in $(d+H)^N$ $(N=0,1,2,\ldots)$ and $(d+H)^{-N}$ $(N=1,2,3,\ldots)$. We have collected each coefficient of these resultant polynomials and setting them to zero, which generates a system of algebraic equations (for convenience the equations are not present here) for a_0, a_1, b_1, d, m and k. We have obtained the following six families of solutions by solving this system of algebraic equations with the help of a symbolic computation software, such as Maple,

Family-1

$$a_0=0,\ a_1=a_1,\ b_1=\frac{a_1(4E\varphi+Q^2)}{4\varphi^2}, d=-\frac{Q}{2\varphi},$$

$$C=-\frac{3(Pa_1)^2}{32\varphi^2}, k=\pm\frac{P}{2\sqrt{-2(4E\varphi+Q^2)}}, \quad \ldots(17)$$

Family-2:

$$a_0=0,\ a_1=0, b_1=b_1, d=-\frac{Q}{2\varphi}, C=-\frac{3(P\varphi b_1)^2}{2(4E\varphi+Q^2)^2},$$

$$k=\pm\frac{P}{\sqrt{-2(4E\varphi+B^2)}},$$

$$m=\pm\frac{P\sqrt{-2(4E\varphi+B^2)\left((4E\varphi+B^2)+16{b_1}^2\varphi^2\right)}}{8b_1\varphi\sqrt{(4E\varphi+B^2)}}, \quad \ldots(18)$$

Family-3:

$$a_0=a_0,\ a_1=a_1,\ b_1=0, d=-\frac{2a_0\varphi+a_1Q}{2\varphi\, a_1}, C=-\frac{3(Pa_1)^2}{32\varphi^2},$$

$$k=-\frac{P}{\sqrt{-2(4E\varphi+Q^2)}},\ m=\pm\frac{P\sqrt{-2({a_1}^2(4E\varphi+Q^2)+\varphi^2)}}{2\,a_1(4E\varphi+Q^2)}, \quad \ldots(19)$$

Family-4:

$$a_0=a_0,\ a_1=0,\ b_1=\frac{2\,a_0(d^2\varphi+dQ-E)}{2d\varphi+Q}, d=-\frac{Q}{2\varphi},$$

$$C=-\frac{3(Pa_0)^2}{8(2d\varphi+Q)^2},\ k=\pm\frac{P}{\sqrt{-2(4E\varphi+Q^2)}},$$

$$m=\pm\frac{P\sqrt{-2(4E\varphi+Q^2)(4\,{a_0}^2+1)-8d^2}}{4\,a_0(4E\varphi+Q^2)}, \quad \ldots(20)$$

Family-5:

$$a_0=0,\ a_1=a_1,\ b_1=\frac{a_1(4E\varphi+Q^2)}{4\varphi^2}, d=-\frac{Q}{2\varphi}, C=0,$$

$$k=\pm\frac{P}{\sqrt{2(4E\varphi+Q^2)}},\ m=\pm\frac{P\sqrt{-2(4\,{a_1}^2(4E\varphi+Q^2)-Q^2)}}{4\,a_1(4E\varphi+Q^2)}, \quad \ldots(21)$$

Family-6:

$$a_0=a_0,\ a_1=0, d=-\frac{\left(Q-\sqrt{(4E\varphi+Q^2)}\right)}{2\varphi}, m=\pm\frac{P}{2\sqrt{(4E\varphi+Q^2)}},$$

$$C=0, k=\pm\frac{P}{2\sqrt{(4E\varphi+Q^2)}},$$

$$b_1=\frac{(4E\varphi+Q^2)a_0}{\varphi\left[\left(-\frac{P}{(4E\varphi+Q^2)}\left(Q-\sqrt{(4E\varphi+Q^2)}\right)\right)+\frac{Q}{(4E\varphi+Q^2)}\left(Q-\sqrt{(4E\varphi+Q^2)}\right)+Q\right]}, \quad \ldots(22)$$

where $\varphi=P-R, a_1, P, Q, R, E$ and d are free parameters.

For simplicity we have investigated solutions only for the family 1 arranged in equation (17) and solutions for other families are omitted here.

When $Q\neq 0, \varphi=P-R$ and $\Omega=Q^2+4E(P-R)>0$, inserting the values of the constants arranged in equation (17) into equation (16) and simplifying, we have obtained the travelling wave solutions:

$$v(\xi)=a_1\left(\frac{\sqrt{\Omega}}{2\varphi}\,\frac{C_1 sinh\left(\frac{\sqrt{\Omega}}{2P}\xi\right)+C_2 cosh\left(\frac{\sqrt{\Omega}}{2P}\xi\right)}{C_1 cosh\left(\frac{\sqrt{\Omega}}{2P}\xi\right)+C_2 sinh\left(\frac{\sqrt{\Omega}}{2P}\xi\right)}\right)+$$

$$\frac{a_1(4E\varphi+Q^2)}{4\varphi^2}\left(\frac{\sqrt{\Omega}}{2\varphi}\,\frac{C_1 sinh\left(\frac{\sqrt{\Omega}}{2P}\xi\right)+C_2 cosh\left(\frac{\sqrt{\Omega}}{2P}\xi\right)}{C_1 cosh\left(\frac{\sqrt{\Omega}}{2P}\xi\right)+C_2 sinh\left(\frac{\sqrt{\Omega}}{2P}\xi\right)}\right)^{-1} \quad \ldots(23)$$

Using the transformation $u=v^{-2}$, the solution [equation (23)] turns into:

$$u(\xi)=\frac{\left(\frac{\sqrt{\Omega}}{2\varphi}\,\frac{C_1 sinh\left(\frac{\sqrt{\Omega}}{2P}\xi\right)+C_2 cosh\left(\frac{\sqrt{\Omega}}{2P}\xi\right)}{C_1 cosh\left(\frac{\sqrt{\Omega}}{2P}\xi\right)+C_2 sinh\left(\frac{\sqrt{\Omega}}{2P}\xi\right)}\right)}{a_1\left(\frac{\sqrt{\Omega}}{2\varphi}\,\frac{C_1 sinh\left(\frac{\sqrt{\Omega}}{2P}\xi\right)+C_2 cosh\left(\frac{\sqrt{\Omega}}{2P}\xi\right)}{C_1 cosh\left(\frac{\sqrt{\Omega}}{2P}\xi\right)+C_2 sinh\left(\frac{\sqrt{\Omega}}{2P}\xi\right)}\right)^2+\frac{a_1(4E\varphi+B^2)}{4\varphi^2}} \quad \ldots(24)$$

where $\xi=k\left(x-\frac{ct^\alpha}{\Gamma(1+\alpha)}\right)$.

Since C_1 and C_2 are integral constants, one might choose the values arbitrarily. Therefore, if we choose $C_1=0$ but, $C_2\neq 0$ and $C_2=0$ but $C_1\neq 0$, then the solutions in [equation (24)] are simplified respectively as:

$$u_1(x,t)=\frac{\left(\frac{\sqrt{\Omega}}{2\varphi}\,coth\left(\frac{\sqrt{\Omega}}{2P}\,k\left(x-\frac{ct^{\alpha}}{\Gamma(1+\alpha)}\right)\right)\right)}{a_1\left(\frac{\sqrt{\Omega}}{2\varphi}\,coth\left(\frac{\sqrt{\Omega}}{2P}\,k\left(x-\frac{ct^{\alpha}}{\Gamma(1+\alpha)}\right)\right)\right)^2+\frac{a_1(4E\varphi+Q^2)}{4\varphi^2}}$$

$$u_2(x,t)=\frac{\left(\frac{\sqrt{\Omega}}{2\varphi}\,tanh\left(\frac{\sqrt{\Omega}}{2P}\,k\left(x-\frac{ct^{\alpha}}{\Gamma(1+\alpha)}\right)\right)\right)}{a_1\left(\frac{\sqrt{\Omega}}{2\varphi}\,tanh\left(\frac{\sqrt{\Omega}}{2P}\,k\left(x-\frac{ct^{\alpha}}{\Gamma(1+\alpha)}\right)\right)\right)^2+\frac{a_1(4E\varphi+Q^2)}{4\varphi^2}}$$

When $Q\neq 0, \varphi=P-R$ and $\Omega=Q^2+4E(P-R)<0$, inserting the values of the constants arranged in equation (17) into equation (16) and using the result $u=v^{-2}$, and also if we choose $C_1=0$ but $C_2\neq 0$ and $C_2=0$ but $C_1\neq 0$, and simplifying, we have obtained the travelling wave solutions respectively as:

$$u_3(x,t)=\frac{\left(\frac{\sqrt{-\Omega}}{2\varphi}\,cot\left(\frac{\sqrt{-\Omega}}{2P}\,k\left(x-\frac{ct^{\alpha}}{\Gamma(1+\alpha)}\right)\right)\right)}{a_1\left(\frac{\sqrt{-\Omega}}{2\varphi}\,cot\left(\frac{\sqrt{-\Omega}}{2P}\,k\left(x-\frac{ct^{\alpha}}{\Gamma(1+\alpha)}\right)\right)\right)^2+\frac{a_1(4E\varphi+Q^2)}{4\varphi^2}}$$

$$u_4(x,t)=\frac{\left(-\frac{\sqrt{-\Omega}}{2\varphi}\,tan\left(\frac{\sqrt{-\Omega}}{2P}\,k\left(x-\frac{ct^{\alpha}}{\Gamma(1+\alpha)}\right)\right)\right)}{a_1\left(-\frac{\sqrt{-\Omega}}{2\varphi}\,tan\left(\frac{\sqrt{-\Omega}}{2P}\,k\left(x-\frac{ct^{\alpha}}{\Gamma(1+\alpha)}\right)\right)\right)^2+\frac{a_1(4E\varphi+Q^2)}{4\varphi^2}}$$

When $Q\neq 0, \varphi=P-R$ and $\Omega=Q^2+4E(P-R)=0$, inserting the values of the constants arranged in equation (17) into equation (16) and using the result $u=v^{-2}$ and also if we choose $C_1=0$ but $C_2\neq 0$ and simplifying, we have obtained the travelling wave solutions respectively as:

$$u_5(\xi)=\frac{\left(\frac{1}{k}\left(x-\frac{ct^{\alpha}}{\Gamma(1+\alpha)}\right)^{-1}\right)}{a_1\left(\frac{1}{k}\left(x-\frac{ct^{\alpha}}{\Gamma(1+\alpha)}\right)^{-1}\right)^2+\frac{a_1(4E\varphi+Q^2)}{4\varphi^2}}$$

However, if we choose $C_2=0$ but $C_1\neq 0$, this leads to a trivial solution, which is not recorded here.

When $Q=0, \varphi=P-R$ and $\Delta=\varphi\,E>0$, inserting the values of the constants arranged in equation (17) into equation (16) and using the result $u=v^{-2}$; and also if we choose $C_1=0$ but $C_2\neq 0$ and choose $C_2=0$ but $C_1\neq 0$, and simplifying, we have obtained the travelling wave solutions respectively as:

$$u_6(x,t)=\frac{\left(\frac{\sqrt{\Delta}}{\varphi}\,coth\left(\frac{\sqrt{\Delta}}{P}\,k\left(x-\frac{ct^{\alpha}}{\Gamma(1+\alpha)}\right)\right)\right)}{a_1\left(\frac{\sqrt{\Delta}}{\varphi}\,coth\left(\frac{\sqrt{\Delta}}{P}\,k\left(x-\frac{ct^{\alpha}}{\Gamma(1+\alpha)}\right)\right)\right)^2+\frac{a_1E}{\varphi}}$$

$$u_7(x,t)=\frac{\left(\frac{\sqrt{\Delta}}{\varphi}\,tanh\left(\frac{\sqrt{\Delta}}{P}\,k\left(x-\frac{ct^{\alpha}}{\Gamma(1+\alpha)}\right)\right)\right)}{a_1\left(\frac{\sqrt{\Delta}}{\varphi}\,tanh\left(\frac{\sqrt{\Delta}}{P}\,k\left(x-\frac{ct^{\alpha}}{\Gamma(1+\alpha)}\right)\right)\right)^2+\frac{a_1E}{\varphi}}$$

When $Q=0, \varphi=P-R$ and $\Delta=\varphi\,E<0$, inserting the values of the constants arranged in equation (17) into equation (16) and using the result $u=v^{-2}$, and also if we choose $C_1=0$ but $C_2\neq 0$ and choose $C_2=0$ but $C_1\neq 0$, and simplifying, we have obtained the travelling wave solutions respectively as:

$$u_8(x,t)=\frac{\left(\frac{\sqrt{-\Delta}}{\varphi}\,cot\left(\frac{\sqrt{-\Delta}}{P}\,k\left(x-\frac{ct^{\alpha}}{\Gamma(1+\alpha)}\right)\right)\right)}{a_1\left(\frac{\sqrt{-\Delta}}{\varphi}\,cot\left(\frac{\sqrt{-\Delta}}{P}\,k\left(x-\frac{ct^{\alpha}}{\Gamma(1+\alpha)}\right)\right)\right)^2+\frac{a_1E}{\varphi}}$$

$$u_9(x,t)=\frac{\left(-\frac{\sqrt{-\Delta}}{\varphi}\,tan\left(\frac{\sqrt{-\Delta}}{P}\,k\left(x-\frac{ct^{\alpha}}{\Gamma(1+\alpha)}\right)\right)\right)}{a_1\left(-\frac{\sqrt{-\Delta}}{\varphi}\,tan\left(\frac{\sqrt{-\Delta}}{P}\,k\left(x-\frac{ct^{\alpha}}{\Gamma(1+\alpha)}\right)\right)\right)^2+\frac{a_1E}{\varphi}}$$

where $k=\pm\frac{P}{2\sqrt{-2(4E\varphi+Q^2)}}$.

It is remarkable to observe that the travelling wave solutions of the time fractional Boussinesq-type equation $u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8$ and u_9 are the new generalised solutions and have not been found in previous literature.

The (3, 3, 3) time fractional Zakharov-Kuznetsov equation

In this sub-section, we have determined some new close form travelling wave solutions to the fractional Zakharov-Kuznetsov equation by using the generalised (G'/G)-method. Let us consider the time fractional ZK equation of the form:

$$D_t^{\alpha} v + a(v^p)_x + b(v^q)_{xxx} + c(v^r)_{xyy} = 0;\ 0 < \alpha \le 1,$$

where α is parameter of fractional order derivative, a, b, c are arbitrary constants and p, q, r are integers and $p, q, r \neq 0$.

To obtain the travelling wave solutions to the above equation, we have investigated the time fractional ZK equation for the case (3, 3, 3):

$$D_t^{\alpha} v + (v^3)_x + 2(v^3)_{xxx} + 2(v^3)_{xyy} = 0;\ 0 < \alpha \le 1, \quad \text{...(25)}$$

This equation describes ion-acoustic waves in plasma, viscoelasticity waves, sound waves, signal processing waves through optical fibers, etc. It also appears in the fields of electromagnetic field, material science, probability statistics, chemical physics and financial mathematics. By means of the traveling transformation [equation (2)], equation (25) is transformed into the following nonlinear ODE:

$$mv' + k(v^3)' + 2k(k^2 + n^2)(v^3)''' = 0, \quad \text{...(26)}$$

which implies that

$$mv + k v^3 + 2k(k^2 + n^2)(v^3)'' + r = 0, \quad \text{...(27)}$$

where r is an integrating constant.

Now, considering the balance between the highest order linear and nonlinear terms occurring in equation (27), yields $N = -1$. Therefore, we have employed a new transformation $v = w^{-1}$, then equation (27) turns into the following nonlinear ODE:

$$m w^4 + k w^2 + 24k(k^2 + n^2)(w')^2 - 6k(k^2 + n^2) w w'' + C = 0, \quad \text{...(28)}$$

Now, balancing the two highest order nonlinear terms occurring in equation (28), yields $N = 1$. Then the solution of equation (28) is the form:

$$w = b_1(d + H)^{-1} + a_0 + a_1(d + H), \quad \text{...(29)}$$

where a_0, a_1, b_1 and d are arbitrary constants to be determined, such that either a_1 or b_1 may be zero, but both a_1 and b_1 cannot be zero at a time.

Substituting equation (29) together with equations (5) and (6) into equation (28), the left hand side is converted into polynomial in $(d + H)^N$ $(N = 0,1,2, \dots)$ and $(d + H)^{-N}$ $(N = 1,2,3, \dots)$. We have collected each coefficient of these resultant polynomials and setting them zero yields a set of simultaneous algebraic equations (for convenience, the equations are not presented here) for a_0, a_1, b_1, d, n, k and m. Solving these algebraic equations with the help of a symbolic computation software such as Maple, we have obtained the following five families of solutions:

Family-1:

$$a_0 = a_0, a_1 = 0, b_1 = -\frac{2a_0(d^2\varphi + Qd - E)}{2d\varphi + Q}, d = d,$$

$$m = \mp \frac{(2d\varphi + Q)^2 \sqrt{(4E\varphi + Q^2)(P^2 - 9n^2(4E\varphi + Q^2))}}{9(4E\varphi + Q^2)^2 a_0{}^2},$$

$$k = \pm \frac{\sqrt{(4E\varphi + Q^2)(P^2 - 9n^2(4E\varphi + Q^2))}}{(4E\varphi + Q^2)}, n = n,$$

$$C = \mp \frac{2a_0{}^2 \sqrt{(4E\varphi + Q^2)(P^2 - 9n^2(4E\varphi + Q^2))}}{9(2d\varphi + Q)^2}, \quad \text{...(30)}$$

Family-2:

$$a_0 = 0, a_1 = a_1, b_1 = \frac{a_1(4E\varphi + Q^2)}{4\varphi^2},$$

$$m = \pm \frac{\varphi^2 \sqrt{(4E\varphi + Q^2)(P^2 - 36n^2(4E\varphi + Q^2))}}{18(4E\varphi + Q^2)^2 a_1{}^2}, d = -\frac{Q}{2\varphi},$$

$$k = \pm \frac{\sqrt{(4E\varphi + Q^2)(P^2 - 36n^2(4E\varphi + Q^2))}}{6(4E\varphi + Q^2)}, n = n,$$

$$C = \mp \frac{2a_1{}^2 \sqrt{(4E\varphi + Q^2)(P^2 - 36n^2\theta)}}{18\varphi^2}, \quad \text{...(31)}$$

Family-3

$$a_0 = 0, a_1 = 0, b_1 = b_1, d = -\frac{Q}{2R},$$

$$m = \mp \frac{\sqrt{(4E\varphi + Q^2)(P^2 - 9n^2(4E\varphi + Q^2))}}{36\varphi^2 b_1{}^2}, n = n,$$

$$k = \pm \frac{\sqrt{(4E\varphi + Q^2)(P^2 - 9n^2(4E\varphi + Q^2))}}{(4E\varphi + Q^2)},$$

$$C = \mp \frac{5b_1{}^2 \varphi^2 \sqrt{(4E\varphi + Q^2)(P^2 - 4n^2\theta)}}{2\theta^2}, \quad \text{...(32)}$$

Family-4:

$a_0 = 0, a_1 = a_1, b_1 = 0, d = d,$

$$m = \mp \frac{4\varphi^2\sqrt{(4E\,\varphi+Q^2)(P^2-9\,n^2(4E\,\varphi+Q^2))}}{9(4E\,\varphi+Q^2)^2 {a_1}^2}, n = n,$$

$$k = \pm \frac{\sqrt{(4E\,\varphi+Q^2)(P^2-9\,n^2(4E\,\varphi+Q^2))}}{3(4E\,\varphi+Q^2)},$$

$$C = \mp \frac{{a_1}^2\sqrt{(4E\,\varphi+Q^2)(P^2-9\,n^2(4E\,\varphi+Q^2))}}{18\,\varphi^2}, \qquad ...(33)$$

Family-5:

$a_0 = 0, a_1 = a_1, b_1 = -\frac{(4E\,\varphi+Q^2)a_1}{4\varphi^2},$

$$m = \mp \frac{2\varphi^2\sqrt{-2(4E\,\varphi+Q^2)(P^2+18\,n^2(4E\,\varphi+Q^2))}}{18(4E\,\varphi+Q^2)^2 {a_1}^2},$$

$C = 0, d = -\frac{Q}{2\varphi}, n = n,$

$$k = \pm \frac{\sqrt{-2(4E\,\varphi+Q^2)(P^2+18n^2(4E\,\varphi+Q^2))}}{6(4E\,\varphi+Q^2)}, \qquad ...(34)$$

where $\varphi = P - R, P, Q, R, d, E$ are free parameters.

For simplicity we have used only the family 1 arranged in equation (30) and the solutions for other families are omitted here.

When $Q \neq 0, \varphi = P - R$ and $\Omega = Q^2 + 4E(P - R) > 0$, inserting the values of the constants arranged in equation (30) into equation (29) and simplifying, we obtained the wave solutions as:

$w(\xi) =$

$$a_0 - \frac{2a_0\left(d^2\varphi+Qd-E\right)}{2d\varphi+Q}\left(d + \frac{Q}{2\,\varphi} + \frac{\sqrt{\Omega}}{2\varphi}\,\frac{C_1 sinh\left(\frac{\sqrt{\Omega}}{2P}\xi\right)+C_2 cosh\left(\frac{\sqrt{\Omega}}{2P}\xi\right)}{C_1 cosh\left(\frac{\sqrt{\Omega}}{2P}\xi\right)+C_2 sinh\left(\frac{\sqrt{\Omega}}{2P}\xi\right)}\right)^{-1} \qquad ..(35)$$

However, when using the result $v = w^{-1}$, the solution (35) obtained is:

$$v(\xi) = \frac{\left(d+\frac{Q}{2\,\varphi}+\frac{\sqrt{\Omega}}{2\varphi}\frac{C_1 sinh\left(\frac{\sqrt{\Omega}}{2P}\xi\right)+C_2 cosh\left(\frac{\sqrt{\Omega}}{2P}\xi\right)}{C_1 cosh\left(\frac{\sqrt{\Omega}}{2P}\xi\right)+C_2 sinh\left(\frac{\sqrt{\Omega}}{2P}\xi\right)}\right)}{a_0\left(d+\frac{Q}{2\,\varphi}+\frac{\sqrt{\Omega}}{2\varphi}\frac{C_1 sinh\left(\frac{\sqrt{\Omega}}{2P}\xi\right)+C_2 cosh\left(\frac{\sqrt{\Omega}}{2P}\xi\right)}{C_1 cosh\left(\frac{\sqrt{\Omega}}{2P}\xi\right)+C_2 sinh\left(\frac{\sqrt{\Omega}}{2P}\xi\right)}\right)-\frac{2a_0(d^2\varphi+Qd-E)}{2d\varphi+Q}} \qquad ...(36)$$

where $\xi = k\left(x + y - \frac{ct^\alpha}{\Gamma(1+\alpha)}\right)$

Since C_1 and C_2 are integral constants, we might choose the values arbitrarily. If we choose $C_1 = 0$ but $C_2 \neq 0$ and $C_2 = 0$ but $C \neq 0$, then the solutions [equation (36)] are simplified respectively as:

$v_1(x, y, t) =$

$$\frac{\left(d+\frac{Q}{2\,\varphi}+\frac{\sqrt{\Omega}}{2\varphi}\,coth\left(\frac{\sqrt{\Omega}}{2P}\,k\left(x+y-\frac{ct^\alpha}{\Gamma(1+\alpha)}\right)\right)\right)}{a_0\left(d+\frac{Q}{2\,\varphi}+\frac{\sqrt{\Omega}}{2\,\varphi}\,coth\left(\frac{\sqrt{\Omega}}{2P}\,k\left(x+y-\frac{ct^\alpha}{\Gamma(1+\alpha)}\right)\right)\right)-\frac{2a_0(d^2\varphi+Qd-E)}{2d\varphi+Q}}$$

$v_2(x, y, t) =$

$$\frac{\left(d+\frac{Q}{2\,\varphi}+\frac{\sqrt{\Omega}}{2\varphi}\,tanh\left(\frac{\sqrt{\Omega}}{2P}\,k\left(x+y-\frac{ct^\alpha}{\Gamma(1+\alpha)}\right)\right)\right)}{a_0\left(d+\frac{Q}{2\,\varphi}+\frac{\sqrt{\Omega}}{2\,\varphi}\,tanh\left(\frac{\sqrt{\Omega}}{2P}\,k\left(x+y-\frac{ct^\alpha}{\Gamma(1+\alpha)}\right)\right)\right)-\frac{2a_0(d^2\varphi+Qd-E)}{2d\varphi+Q}}$$

When $Q \neq 0$, $\varphi = P - R$ and $\Omega = Q^2 + 4E(P - R) < 0$, inserting the values of the constants arranged in equation (30) into equation (29) and using the result $v = w^{-1}$, and also if we choose $C_1 = 0$ but $C_2 \neq 0$ and choose $C_2 = 0$ but $C_1 \neq 0$., and simplifying, we have attained the travelling wave solutions respectively as:

$v_3(x, y, t) =$

$$\frac{\left(d+\frac{Q}{2\,\varphi}+\frac{\sqrt{-\Omega}}{2\varphi}\,cot\left(\frac{\sqrt{-\Omega}}{2P}\,k\left(x+y-\frac{ct^\alpha}{\Gamma(1+\alpha)}\right)\right)\right)}{a_0\left(d+\frac{Q}{2\,\varphi}+\frac{\sqrt{-\Omega}}{2\,\varphi}\,cot\left(\frac{\sqrt{-\Omega}}{2P}\,k\left(x+y-\frac{ct^\alpha}{\Gamma(1+\alpha)}\right)\right)\right)-\frac{2a_0(d^2\varphi+Qd-E)}{2d\varphi+Q}}$$

$v_4(x, y, t) =$

$$\frac{\left(d+\frac{Q}{2\,\varphi}-\frac{\sqrt{-\Omega}}{2\varphi}\,tan\left(\frac{\sqrt{-\Omega}}{2P}\,k\left(x+y-\frac{ct^\alpha}{\Gamma(1+\alpha)}\right)\right)\right)}{a_0\left(d+\frac{Q}{2\,\varphi}-\frac{\sqrt{-\Omega}}{2\varphi}\,tan\left(\frac{\sqrt{-\Omega}}{2P}\,k\left(x+y-\frac{ct^\alpha}{\Gamma(1+\alpha)}\right)\right)\right)-\frac{2a_0(d^2\varphi+Qd-E)}{2d\varphi+Q}}$$

When $Q \neq 0$, $\varphi = P - R$ and $\Omega = Q^2 + 4E(P - R) = 0$, inserting the values of the constants arranged in equation (30) into equation (29) and using the result $v = w^{-1}$, and also if we choose $C_1 = 0$ but $C_2 \neq 0$ and choose $C_2 = 0$ but $C_1 \neq 0$, and simplifying, we have attained the wave solutions respectively as:

$v_5(x,y,t) =$

$$\frac{\left(d+\frac{Q}{2\varphi}+\frac{1}{k}\left(x+y-\frac{ct^{\alpha}}{\Gamma(1+\alpha)}\right)^{-1}\right)}{a_0\left(d+\frac{Q}{2\varphi}+\frac{1}{k}\left(x+y-\frac{ct^{\alpha}}{\Gamma(1+\alpha)}\right)^{-1}\right)-\frac{2a_0(d^2\varphi+Qd-E)}{2d\varphi+Q}}$$

However, if we choose $C_2 = 0$ but $C_1 \neq 0$, then in this case we get a trivial solution, which is not recorded here.

When $Q = 0$, $\varphi = P - R$ and $\Delta = \varphi E > 0$, inserting the values of the constants arranged in equation (30) into equation (29) and using the result $v = w^{-1}$, and also if we choose $C_1 = 0$ but $C_2 \neq 0$ and choose $C_2 = 0$ but $C_1 \neq 0$, and simplifying, we have attained the travelling wave solutions respectively as:

$v_6(x,y,t) =$

$$\frac{\left(d+\frac{\sqrt{\Delta}}{\varphi}\coth\left(\frac{\sqrt{\Delta}}{P}k\left(x+y-\frac{ct^{\alpha}}{\Gamma(1+\alpha)}\right)\right)\right)}{a_0\left(d+\frac{\sqrt{\Delta}}{\varphi}\coth\left(\frac{\sqrt{\Delta}}{P}k\left(x+y-\frac{ct^{\alpha}}{\Gamma(1+\alpha)}\right)\right)\right)-\frac{a_0(d^2\varphi-E)}{d\varphi}}$$

$v_7(x,y,t) =$

$$\frac{\left(d+\frac{\sqrt{\Delta}}{\varphi}\tanh\left(\frac{\sqrt{\Delta}}{P}k\left(x+y-\frac{ct^{\alpha}}{\Gamma(1+\alpha)}\right)\right)\right)}{a_0\left(d+\frac{\sqrt{\Delta}}{\varphi}\tanh\left(\frac{\sqrt{\Delta}}{P}k\left(x+y-\frac{ct^{\alpha}}{\Gamma(1+\alpha)}\right)\right)\right)-\frac{a_0(d^2\varphi-E)}{d\varphi}}$$

When $Q = 0$, $\varphi = P - R$ and $\Delta = \varphi E < 0$, inserting the values of the constants arranged in equation (30) into equation (29) and using the result $v = w^{-1}$, and also if we choose $C_1 = 0$ but $C_2 \neq 0$ and choose $C_2 = 0$ but $C_1 \neq 0$, and simplifying, we have attained the travelling wave solutions respectively as:

$v_8(x,y,t) =$

$$\frac{\left(d+\frac{\sqrt{-\Delta}}{\varphi}\cot\left(\frac{\sqrt{-\Delta}}{P}k\left(x+y-\frac{ct^{\alpha}}{\Gamma(1+\alpha)}\right)\right)\right)}{a_0\left(d+\frac{\sqrt{-\Delta}}{\varphi}\cot\left(\frac{\sqrt{-\Delta}}{P}k\left(x+y-\frac{ct^{\alpha}}{\Gamma(1+\alpha)}\right)\right)\right)-\frac{a_0(d^2\varphi-E)}{d\varphi}}$$

$v_9(x,y,t) =$

$$\frac{\left(d-\frac{\sqrt{-\Delta}}{\varphi}\tan\left(\frac{\sqrt{-\Delta}}{P}k\left(x+y-\frac{ct^{\alpha}}{\Gamma(1+\alpha)}\right)\right)\right)}{a_0\left(d-\frac{\sqrt{-\Delta}}{\varphi}\tan\left(\frac{\sqrt{-\Delta}}{P}k\left(x+y-\frac{ct^{\alpha}}{\Gamma(1+\alpha)}\right)\right)\right)-\frac{a_0(d^2\varphi-E)}{d\varphi}}$$

where $k = \pm\frac{\sqrt{(4E\varphi+Q^2)(P^2-9n^2(4E\varphi+Q^2))}}{(4E\varphi+Q^2)}$.

It is noteworthy to see that the travelling wave solutions of the (3, 3, 3) time fractional ZK v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8 and v_9 are new generalised solutions, which have not been found in literature.

Graphical representations

In this sub-section, we have discussed the graphical representations of the obtained solutions for 3D and 2D figures by means of symbolic software like Mathematica of the nonlinear time fractional Boussinesq-type and the (3, 3, 3) time fractional ZK equations as follows:

For simplicity the figures related to other solutions $u_2, u_4, u_8, v_1, v_2, v_3, v_5, v_7$ and v_9 are omitted because we have seen more or less the same behaviour like the other solutions.

Physical explanations

In this subsection we have graphically demonstrated different nature of obtained solutions of the non-linear time fractional Boussinesq-type and the (3, 3, 3) time fractional Zakharov-Kuznetsov (FZK) equations. We have displayed different nature such as kink, singular kink, kink type singular periodic and bell type singular periodic solutions. We have depicted 3D and 2D Figures 1 to 8 of solutions $u_1, u_3, u_5, u_6, u_7, u_9, v_4, v_6$ and v_8, respectively for different parameter values. Figures 1 and 8 represent kink type solution of u_1 for $R = Q = E = c = \Gamma = 1$, $a_1 = -1$, $P = 2$, $\alpha = 0.25$ and v_6 for $d = -2$, $Q = 0$, $R = c = E = \Gamma = 1$, $a_1 = -1$, $P = 2$, $\alpha = 0.25$, respectively in the region $-10 \leq x \leq 10$, $0 \leq t \leq 10$. Figures 2, 4, 5 and 8 of solutions u_3 for $R = Q = c = \Gamma = 1$, $a_1 = E = -1$, $P = 2$, $\alpha = 0.25$ u_7 for $Q = d = 0$, $R = c = \Gamma = 1$, $a_1 = E = -1$, $P = 2$, $\alpha = 0.25$, u_9 for $Q = d = 0$, $R = c = \Gamma = 1$, $a_1 = E = -1$, $P = 2$, $\alpha = 0.25$ and v_8 for $d = -2$, $Q = 0$, $R = c = \Gamma = 1$, $a_1 = E = -1$, $P = 2$, $\alpha = 0.25$, respectively shows the kink type singular periodic solutions. We have Also plotted the singular kink wave of Figure 4 for u_6 when $Q = d = 0$, $R = c = E = \Gamma = 1$, $a_1 = -1$, $P = 2$, $\alpha = 0.25$ and Figure 7 is the periodic bell type solution of v_4 for $d = -2$, $Q = R = c = \Gamma = 1$, $a_1 = E = -1$, $P = 2$, $\alpha = 0.25$. All the figures are drawn within the range $-10 \leq x \leq 10$, $0 \leq t \leq 10$.

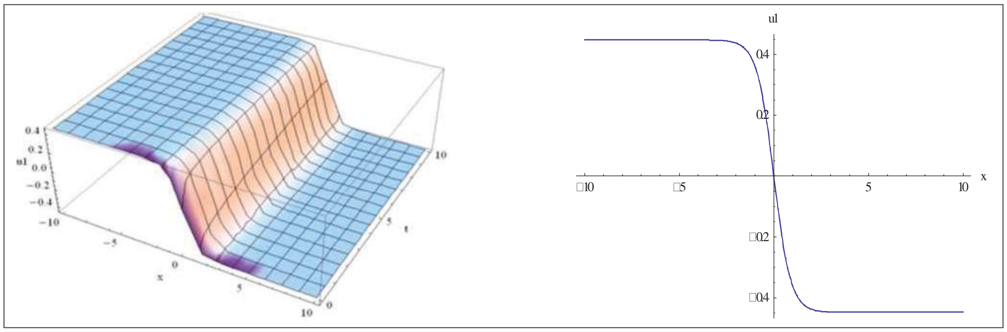

Figure 1: Plot of kink wave of u_1 when $R = Q = E = c = \Gamma = 1$, $P = 2$, $\alpha = 0.25$ and $-10 \leq x \leq 10$, $0 \leq t \leq 10$

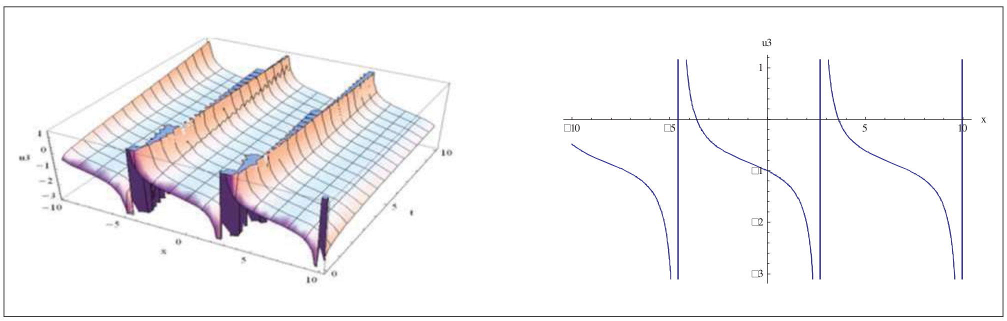

Figure 2: Plot of singular periodic of u_3 when $R = Q = c = \Gamma = 1$, $a_1 = E = -1$, $P = 2$, $\alpha = 0.25$ and $-10 \leq x \leq 10$, $0 \leq t \leq 10$

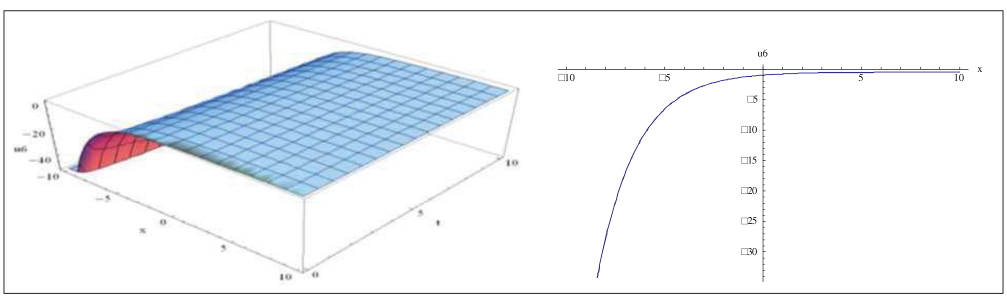

Figure 3: Plot of singular kink wave of u_6 for $Q = d = 0$, $R = c = E = \Gamma = 1$, $a_1 = -1$, $P = 2$, $\alpha = 0.25$ and $-10 \leq x \leq 10$, $0 \leq t \leq 10$

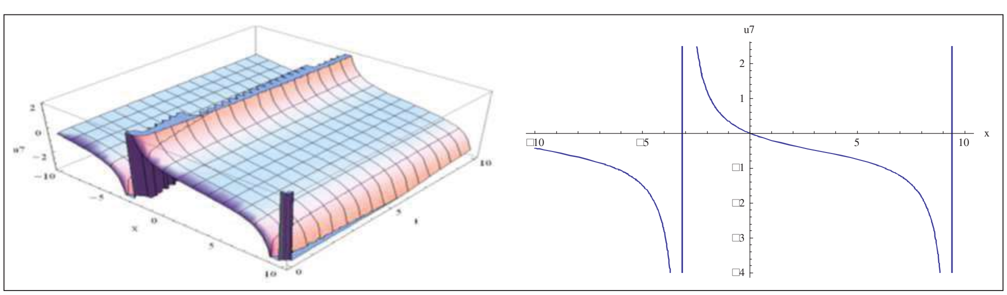

Figure 4: Plot of singular kink wave of u_7 for $Q = d = 0$, $R = c = \Gamma = 1$, $a_1 = E = -1$, $P = 2$, $\alpha = 0.25$ and $-10 \leq x \leq 10$, $0 \leq t \leq 10$.

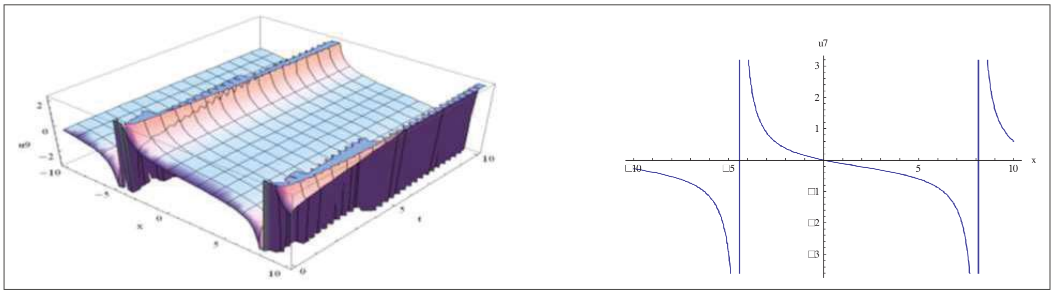

Figure 5: Plot of singular kink wave of u_9 for $Q = d = 0$, $R = c = \Gamma = 1$, $a_1 = E = -1$, $P = 2$, $\alpha = 0.25$ and $-10 \le x \le 10$, $0 \le t \le 10$

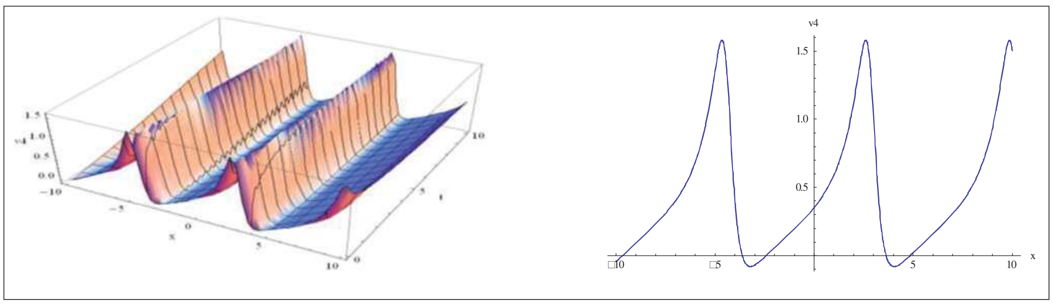

Figure 6: Plot of periodic bell type solution v_4 for $d = -2$, $Q = R = c = \Gamma = 1, a_1 = E = -1$, $P = 2$, $\alpha = 0.25$ and $-10 \le x \le 10$, $0 \le t \le 10$

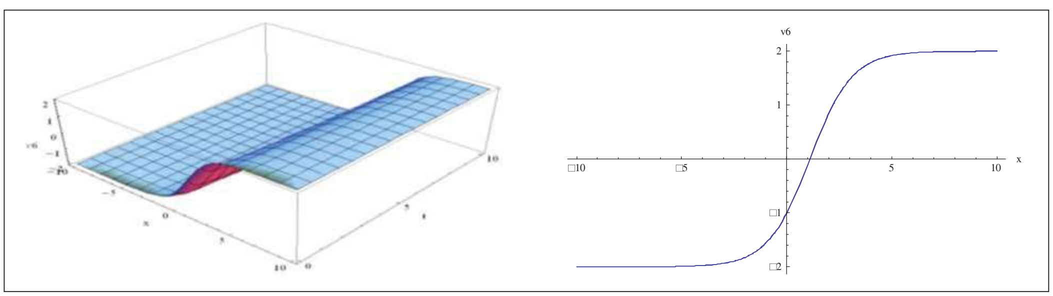

Figure 7: Plot of kink type solution of v_6 for $d = -2$, $Q = 0$, $R = c = E = \Gamma = 1$, $a_1 = -1$, $P = 2$, $\alpha = 0.25$ and $-10 \le x \le 10$, $0 \le t \le 10$

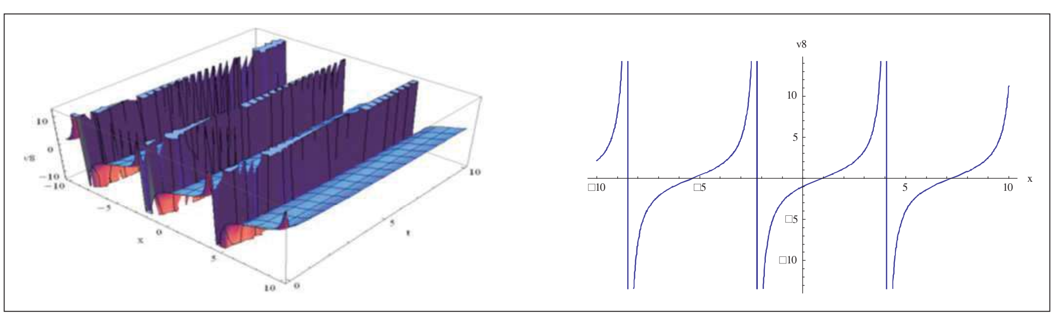

Figure 8: Plot of singular periodic solution v_8 for $d = -2$, $Q = 0$, $R = c = \Gamma = 1$, $a_1 = E = -1$, $P = 2$, $\alpha = 0.25$ and $-10 \le x \le 10$, $0 \le t \le 10$

CONCLUSIONS

In this article, we have determined the new generalised solutions to the time fractional Boussinesq-type equation and the (3, 3, 3) time fractional ZK equation by the efficient and prevailing technique known as the new generalised (G'/G)-expansion method. We have discussed the obtained solutions with different nature and their possible applications. The obtained solutions maybe used to explain the behaviour of fluid flow and heat transfer process, in generally and flow depends on temperature, waves on free moving fluid surface, ion-acoustic wave and traffic flow, in particular.

We have also shown this method generate generalised large number of solutions with several free parameters. By choosing appropriate parameter values obtained solutions maybe utilised to explain many physical phenomena in the field of physics and engineering. Further this method maybe used to find generalised solutions of many time fractional non-linear differential equations appear in the field of mathematical physics.

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