*J.Natn.Sci.Foundation Sri Lanka 2020 48 (3): 265 - 274*
DOI: http://dx.doi.org/10.4038/jnsfsr.v48i3.8836

RESEARCH ARTICLE

# Topp-Leone moment exponential distribution: properties and applications

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Submitted: 27 February 2019; Revised: 14 January 2020; Accepted: 29 May 2020

**Abstract:** In this article, a new three parameter lifetime model is proposed as a generalisation of the moment exponential distribution. The proposed model is named as Topp-Leone moment exponential distribution. The induction of two additional shape parameters will enhance the capability of the proposed model to handle the complex scenarios in modelling. Several properties of the proposed model are discussed. The model parameters are estimated using method of maximum likelihood. Real life applications of the proposed model have been carried out by using datasets from the fields of botany, archaeology and ecology.

**Keywords:** Characterisation, estimation, moment exponential, reliability, Topp-Leone-G family.

## INTRODUCTION

The selection of an appropriate model to analyse the behaviour of real data is an attractive and complicated task. In certain situations, the existing classical models are not suitable to express the utility of real data in several applied areas. These complexities clearly demand useful models to handle these scenarios. A comprehensive discussion is available in the literature (Azzalini, 1985; Freimer *et al*., 1988; Marshall & Olkin, 1997; Eugene *et al*., 2002; Azzalini & Capitanio, 2003; Alzaatreh *et al*., 2013) about the generalisation of classical distributions.

In lifetime data analysis, exponential (Ex) distribution is strongly recommended due to its interesting 'lack of memory' property, which makes it more flexible for modification. The probability density function (PDF) of this distribution is decreasing with a constant hazard rate function (HRF). This limitation opens doors for various generalisations of the exponential distribution. Various extensions of the Ex distribution have been proposed in literature, see for example, the exponentiated Ex (Gupta & Kundu, 2001), extended exponentiated Ex (Abu-Youssef *et al*., 2015), gamma exponentiated Ex (Ristic & Balakrishnan, 2012), beta Ex (Nadarajah & Kotz, 2006), odd exponentiated half logistic Ex (Afify *et al*., 2018) and generalised odd log-logistic Ex (Afify *et al*., 2019), among others.

Dara and Ahmed (2012) proposed a new extension of the Ex distribution called moment exponential (MEx) distribution. The cumulative distribution function (CDF) of the MEx distribution is

$$F(x) = 1 - (1 + x\beta)e^{-\beta x}, \quad x > 0,\ \beta > 0. \qquad \dots (1)$$

The corresponding PDF is

$$f(x) = \beta^2 x e^{-\beta x}, \quad x > 0, \qquad \beta > 0, \qquad \dots (2)$$

where, $\beta$ is a scale parameter. Dara and Ahmed (2012) provided many properties and applications of the MEx

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distribution. Some extensions of the MEx distribution have been proposed in literature, for example, Hasnain (2013) derived the exponentiated MEx distribution and Iqbal *et al*. (2014) developed a new generalisation of exponentiated MEx distribution.

In practice, the most suitable form of hazard rate is the bathtub form. Topp and Leone (1955) introduced a model, which exhibits bathtub hazard rate with a bounded domain on [0,1]. Due to restricted domain, Topp and Leone distribution is not a suitable distribution for modelling of lifetime data. Rezaei *et al*. (2017) constructed a generalised class of distributions based on Topp and Leone distribution called Topp-Leone-G (TL-G) family. The TL-G family provides bathtub-shaped hazard rate.

The CDF and PDF of the TL-G class are given as

$$F(x;\alpha,\gamma,\boldsymbol{\eta}) = [G(x;\boldsymbol{\eta})^{\gamma}\{2 - G(x;\boldsymbol{\eta})^{\gamma}\}]^{\alpha} \qquad \text{...(03)}$$

and

$$f(x;\alpha,\gamma,\boldsymbol{\eta}) = \frac{2\alpha\gamma g(x;\boldsymbol{\eta})}{G(x;\boldsymbol{\eta})^{1-\gamma\alpha}}[1 - G(x;\boldsymbol{\eta})^{\gamma}][2 - G(x;\boldsymbol{\eta})^{\gamma}]^{\alpha-1},$$

$$\alpha,\gamma > 0, \qquad \text{...(04)}$$

where $G(x;\eta)$ is a baseline CDF with parameter vector η, $g(x;\eta)$ is the corresponding baseline PDF, $\alpha$ and $\gamma$ are positive shape parameters.

In this article, we have provided a new generalisation of the MEx distribution using the TL-G family (Rezaei *et al*., 2017). The proposed model is referred to as Topp-Leone moment exponential (TLMEx) distribution. The induction of two shape parameters escalates the adaptability of the TLMEx model. The closed form of its hazard rate makes it more flexible to model life time datasets.

## THE TLMEX DISTRIBUTION

In this section, the CDF and PDF of the TLMEx distribution are defined. The CDF of the TLMEx follows using equations (1) and (3) as

$$F(x;\alpha,\beta,\gamma) = \left(\left[1-(1+\beta x)e^{-\beta x}\right]^{\gamma}\right.$$

$$\left.\left\{2-\left[1-(1+\beta x)e^{-\beta x}\right]^{\gamma}\right\}\right)^{\alpha}, \quad x,\beta,\alpha,\gamma > 0,$$

and the corresponding PDF is

$$f(x;\alpha,\beta,\gamma) = 2\alpha\gamma\beta^{2}xe^{-\beta x}\left[1-(1+\beta x)e^{-\beta x}\right]^{\gamma\alpha-1}$$

$$\left\{1-\left[1-(1+\beta x)e^{-\beta x}\right]^{\gamma}\right\}$$

$$\left\{2-\left[1-(1+\beta x)e^{-\beta x}\right]^{\gamma}\right\}^{\alpha-1},$$

$$x,\beta,\alpha,\gamma > 0, \qquad \text{...(06)}$$

where $\beta$ is a scale parameter and $\alpha$ and $\gamma$ are shape parameters. The plot of density function for various combinitions of parameters is given in Figure 1.

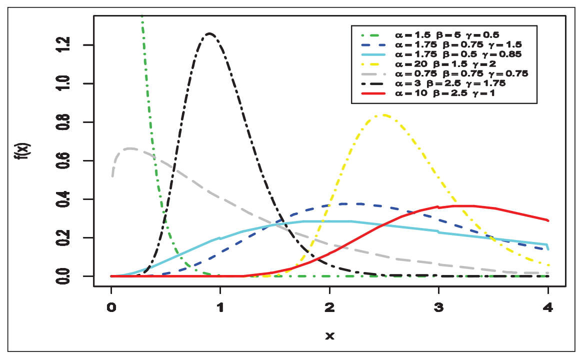


**Figure 1:** Density plots of TLMEx distribution for various values of parameters

Moreover,

$$f(x;\alpha,\beta,\gamma) \sim 2^{\alpha}\gamma\beta^{2}xe^{-\beta x}[1-(1+\beta x)e^{-\beta}\ ]^{\alpha\gamma-1},$$

when $x$ approaches the lower end point of $F(x)$ and when $x$ approaches the upper end point $F(x)$

$$f(x;\alpha,\beta,\gamma) \sim 2\gamma\beta^{2}xe^{-\beta x}\left\{1-\left[1-(1+\beta x)e^{-\beta x}\right]^{\gamma}\right\}.$$

Similarly, for hazard rate, $h(x;\alpha,\beta,\gamma) \sim 2^{\alpha}\gamma\beta^{2}xe^{-\beta x}$ $[1-(1+x\beta)e^{-\beta x}]^{\alpha\gamma-1}$ when $x$ approaches the lower end point of $F(x)$ and when $x$ approaches the upper end point of $F(x)$, $h(x) \sim 2\gamma\beta 2xe^{-\beta x}$. These approximations help to derive the bathtub shapes of the HRF.

The HRF of the TLMEx model in general is

$$h(x;\alpha,\beta,\gamma) =$$

$$\frac{2\alpha\gamma\beta^2 x e^{-\beta x}\left[1-(1+\beta x)e^{-\beta x}\right]^{\gamma\alpha-1}\left\{1-\left[1-(1+\beta x)e^{-\beta x}\right]^{\gamma}\right\}}{1-\left([1-(1+\beta x)e^{-\beta x}]^{\gamma}\{2-[1-(1+\beta x)e^{-\beta}\ ]^{\gamma}\}\right)^{\alpha}}$$

$$\times\left\{2-\left[1-(1+\beta x)e^{-\beta x}\right]^{\gamma}\right\}^{\alpha-1}.$$

Figure 2 provides some shapes of the TLMEx HRF for different choices of parameters. The HRF of the TLMEx distribution can be constant, decreasing, increasing or bathtub failure rate shape.

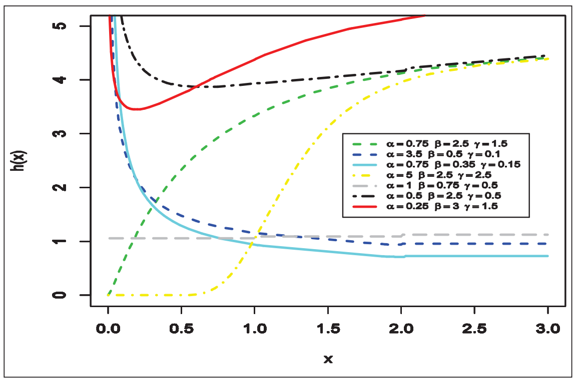


**Figure 2:** HRF plots of TLMEx distribution for various values of parameters

The PDF of the TLMEx distribution can be expressed in a mixture form as

$$f(x) = \sum_{j,i=0}^{\infty} \delta_{j,i}\, \beta^{i+2} x^{i+1} e^{-(j+1)\beta x}, \qquad \dots (7)$$

where $\delta_{j,i}$ is the constant term given by

$$\delta_{j,i} = \sum_{k=0}^{\infty} (-1)^{k+j} 2^{\alpha-k} \gamma(\alpha+k) \binom{\alpha}{k}\binom{j}{i}\binom{\gamma(\alpha+k)-1}{j}.$$

Based on equation (7), many statistical properties of the TLMEx model can be easily studied.

## Mathematical characteristics

In this section, some mathematical properties of the TLMEx distribution are discussed. These include moments, moment generating function, reliability analysis, Lorenz curve and a certain characterisation.

### Moments

Moments are the fundamental property for any distribution function. The $qth$ raw moment of a distribution is defined as

$$E(X^q) = \int_{-\infty}^{\infty} x^q dF(x).$$

Using the PDF equation (7) and assuming $z = (j+1)\beta x$, one can write

$$E(X^q) = \sum_{j,i=0}^{\infty} \delta_{j,i} \beta^{-q} (j+1)^{-(q+i+2)} \Gamma(q+i+2).$$

The mean of the TLMEx distribution is easily obtained by setting $q=1$, and is

$$E(X) = \sum_{j,i=0}^{\infty} \delta_{j,i} \beta^{-1} (j+1)^{-(i+3)} \Gamma(i+3).$$

The moment generating function (MGF) of X is defined by

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x) dx.$$

Using $e^{tx} = \sum_{p=0}^{\infty} \frac{(tx)^p}{p!}$ and equation (7), the MGF of the TLMEx distribution reduces to

$$M_X(t) = \sum_{j,i,p=0}^{\infty} \frac{y!}{p!} \delta_{j,i} \beta^{-q} (j+1)^{-(q+i+2)} \Gamma(q+i+2).$$

The moment generating function provides all the moments of the distribution.

### Residual life and reversed residual life

The $n$th moment of the residual life is defined by

$$\pi_n(t) = E[(X-t)^n | X > t] = \frac{1}{R(t)} \int_t^{\infty} (x-t)^n f(x) dx,$$

$$n = 1,2,3,\dots$$

After some simplifications, the above expression reduces to

$$\pi_n(t) = \frac{1}{R(t)} \sum_{k,r=0}^{\infty} (-t)^{n-r} t_k \binom{n}{r} \int_t^{\infty} x^r f(x) dx.$$

Using equation (7) to solve the $\int_t^{\infty} x^r f(x) dx$ and after some simplifications, we have

$$\pi_n(t) = \frac{1}{R(t)} \sum_{k,r,i,j=0}^{\infty} (-t)^{n-r} t_k \binom{n}{r} \delta_{j,i} \frac{1}{\beta^r} \Gamma(\gamma+i+2,$$

$$\beta x(j+1)).$$

We can find the mean residual life (MRL) from the above equation by replacing $n = 1$. The MRL is the expected additional life length for a unit which is alive at age $t$.

The $n$th moment of reserved residual life is given by

$$\kappa_n(t) = E[(t-X)^n | X \le t] = \frac{1}{F(t)} \int_0^t (x-t)^n f(x)dx,$$

$$n = 1,2,3,\ldots$$

We can write

$$\kappa_n(t) = \frac{1}{R(t)} \sum_{k,r=0}^{\infty} (-1)^r t^{n-r} t_k \binom{n}{r} \int_0^t x^r f(x)dx.$$

Using equation (7), the $n$th moment of the reversed residual life of the TLMEx distribution is

$$\kappa_n(t) = \frac{1}{R(t)} \sum_{k,r,j,i=0}^{\infty} (-1)^r t^{n-r} t_k \binom{n}{r} \delta_{j,i} \frac{1}{\beta^r} \gamma(\gamma + i + 2, \beta x(j+1)).$$

**Lorenz curve**

The Lorenz curve (LC) is considered to explain the graph of ratio for any positive random variable in favour of $F(t)$ with the property $L(p) \le p$, $L(0) = 0$, and $L(1) = 1$. The LC is defined as

$$L(F(t)) = \frac{E(X|X\le t)}{E(X)} = \frac{\int_0^t x f(x)dx}{\int_{-\infty}^{\infty} x f(x)dx}.$$

The LC is used to model the income data. Some notable applications of LC can be found in Lindley (1958). Using equation (7), we have

$$\int_0^t x f(x)dx = \sum_{j,i=0}^{\infty} \delta_{j,i} \frac{1}{\beta^q} \left(\frac{1}{j+1}\right)^{i+3} \gamma(i+3, \beta x(j+1)).$$

The LC for the TLMEx model becomes

$$L(p) = \frac{\sum_{j,i=0}^{\infty} \delta_{j,i} \frac{1}{\beta^q} \left(\frac{1}{j+1}\right)^{i+3} \gamma(i+3, \beta x(j+1))}{\sum_{j,i=0}^{\infty} \delta_{j,i} \frac{1}{\beta^q} \left(\frac{1}{j+1}\right)^{q+i+2} \Gamma(q+i+2)}.$$

**Characterization**

In this section, we provide a characterisation of the concomitant of the record statistics for $X$.

**Theorem:** If $X$ is an absolutely continuous random variable and its $n$th moment exist, then

$$E(X^n | X \le w) = A^* \int_0^w x^n f(x)dx.$$

if

$$F(w) = \left(\left[1-(1+\beta w)e^{-\beta w}\right]^{\gamma} \left\{2 - \left[1-(1+\beta w)e^{-\beta w}\right]^{\gamma}\right\}\right)^{\alpha},$$

where

$$A^* = \frac{1}{\left(\left[1-(1+\beta w)e^{-\beta w}\right]^{\gamma} \{2 - \left[1-(1+\beta w)e^{-\beta w}\right]^{\gamma}\}\right)^{\alpha}}$$

and

$$f(x) = 2\alpha\gamma\beta^2 x e^{-t\beta} \left[1-(1+\beta x)e^{-\beta x}\right]^{\gamma\alpha-1} \left\{1 - \left[1-(1+\beta x)e^{-\beta x}\right]^{\gamma}\right\} \left\{2 - \left[1-(1+\beta x)e^{-\beta x}\right]^{\gamma}\right\}^{\alpha-1}.$$

**Proof:** We have

$$E(X^n | X \le w) = \int_0^w x^n \frac{f(x)}{F(w)} dx.$$

Then

$$\int_0^w x^n \frac{f(x)}{F(w)} dx = A^* \int_0^w t^n f(t)dt.$$

$$\int_0^w x^n f(x)dx = F(w) A^* \int_0^w x^n f(x)dx. \quad \ldots (8)$$

Differentiating both sides of equation (8) with respect to $w$, we get

$$w^n f(w) = f(w) A^* \int_0^w x^n f(x)dx + F(w) a^* \int_0^w x^n f(x)dx + F(w) A^* w^n M^*,$$

where

$$a^* = -\frac{2\alpha\gamma\beta^2 w e^{-\beta w} \left\{1 - \left[1-(1+\beta w)e^{-\beta w}\right]^{\gamma}\right\}}{\left([1-(1+\beta w)e^{-\beta w}]^{\gamma}\{2-[1-(1+\beta w)e^{-\beta w}]^{\gamma}\}\right)^{\alpha+1}}$$

and

$$M^* = 2\alpha\gamma\beta^2 w e^{-\beta w} \left[1-(1+\beta w)e^{-\beta w}\right]^{\gamma\alpha-1} \left\{1 - \left[1-(1+\beta w)e^{-\beta w}\right]^{\gamma}\right\} \times \left\{2 - \left[1-(1+\beta w)e^{-\beta w}\right]^{\gamma}\right\}^{\alpha-1}.$$

After some simplifications, the above expression reduces to

$$f(w)\left[w^n - A^* \int_0^w x^n f(x)dx\right] = F(w) \frac{M^*}{A^*} \left[w^n - A^* \int_0^w x^n f(x)dx\right].$$

Then

$$\frac{f(w)}{F(w)} = \frac{M^*}{A^*}. \quad \ldots (9)$$

On integrating the last expression with respect to $w$, we obtain

$$-\ln(F(x)) = -\ln(B^*),$$

where

$$B^* = \left(\left[1-(1+\beta x)e^{-\beta x}\right]^{\gamma}\{2-\left[1-(1+\beta x)e^{-\beta x}\right]^{\gamma}\}\right)^{\alpha}.$$

Thus

$$F(x) = \left(\left[1-(1+\beta x)e^{-\beta}\ \right]^{\gamma}\{2-\left[1-(1+\beta x)e^{-\beta x}\right]^{\gamma}\}\right)^{\alpha},\quad 0 \le x.$$

**Estimation**

This section is devoted to maximum likelihood estimation of parameters of the TLMEx distribution. For this suppose $x_1, \ldots\ldots, x_2$ be a random sample from this distribution with a parameter vector $\Theta = (\alpha, \beta, \gamma)^T$. The log-likelihood function of $\Theta$ is immediately written as

$$\ell = n\log(2\alpha\gamma\beta^2) + \sum_{i=1}^{n}\log(x_i) - \beta\sum_{i=1}^{n}x_i + (\alpha\gamma-1)\sum_{i=1}^{n}\log(S_i) + \sum_{i=1}^{n}\log(1-S_i{}^{\gamma}) + (\alpha-1)\sum_{i=1}^{n}\log(2-S_i{}^{\gamma}),$$

where $S_i = 1-(1+\beta x_i)e^{-\beta x_i}$.

The components of the score vector are

$$\frac{\partial\ell}{\partial\alpha} = \frac{n}{\alpha} + \gamma\sum_{i=1}^{n}\log(S_i) + \sum_{i=1}^{n}\log(2-S_i{}^{\gamma}),$$

$$\frac{\partial\ell}{\partial\beta} = \frac{2n}{\beta} - \sum_{i=1}^{n}x_i + (\alpha\gamma-1)\sum_{i=1}^{n}\frac{\beta K_i}{S_i} - \sum_{i=1}^{n}\frac{\beta\gamma K_i S_i{}^{\gamma-1}}{1-S_i{}^{\gamma}} - (\alpha-1)\sum_{i=1}^{n}\frac{\beta\gamma K_i S_i{}^{\gamma-1}}{2-S_i{}^{\gamma}}$$

and

$$\frac{\partial\ell}{\partial\gamma} = \frac{n}{\gamma} + \alpha\sum_{i=1}^{n}\log(S_i) - \sum_{i=1}^{n}\frac{S_i{}^{\gamma}\log(S_i)}{1-S_i{}^{\gamma}} - (\alpha-1)\sum_{i=1}^{n}\frac{S_i{}^{\gamma}\log(S_i)}{2-S_i{}^{\gamma}},$$

where $K_i = x_i^2 e^{-\beta x_i}$.

The MLEs of the $\Theta = (\alpha, \beta, \gamma)^T$ can be obtained by equating the above nonlinear system of equations to zero and solving them simultaneously. This can be done by using nonlinear optimisation methods such as the quasi Newton algorithm to maximize $\ell$ numerically. The entries of the Fisher information matrix are given in Appendix A.

## SIMULATION RESULTS

In this section, a simulation study is conducted to assess the performance of the MLEs of the TLMEx parameters.

**Table 1:** The average estimates of the parameters and their SEs

| $(\alpha,\beta,\gamma)$ | $n$ | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\gamma}$ | $SE(\hat{\alpha})$ | $SE(\hat{\beta})$ | $SE(\hat{\gamma})$ |
|---|---|---|---|---|---|---|---|
| (1.5, 2.5, 1.5) | 20 | 1.4991 | 2.4980 | 1.4997 | 0.0685 | 0.0488 | 0.0241 |
| | 30 | 1.4969 | 2.4997 | 1.5010 | 0.0628 | 0.0475 | 0.0193 |
| | 50 | 1.5002 | 2.5017 | 1.4998 | 0.0294 | 0.0461 | 0.0168 |
| | 100 | 1.4989 | 2.5002 | 1.5003 | 0.0252 | 0.0461 | 0.0127 |
| | 200 | 1.4997 | 2.4971 | 1.4999 | 0.0222 | 0.0138 | 0.0085 |
| (2.5, 1.5, 3.5) | 20 | 2.4977 | 1.5020 | 3.4998 | 0.0702 | 0.0350 | 0.0208 |
| | 30 | 2.5031 | 1.4974 | 3.5005 | 0.0585 | 0.0327 | 0.0194 |
| | 50 | 2.4979 | 1.5012 | 3.5012 | 0.0425 | 0.0252 | 0.0182 |
| | 100 | 2.5000 | 1.5001 | 3.5003 | 0.0372 | 0.0238 | 0.0160 |
| | 200 | 2.5010 | 1.5004 | 3.5006 | 0.0337 | 0.0213 | 0.0096 |

The simulation study is conducted by drawing random samples of different sizes from this distribution. Since the CDF is not easily invertible, a numerical method of solving nonlinear equation $F(x) = u$, is used, where u is a uniform random number, for various choices of the parameters as discussed by Lange (2010). Maximum likelihood estimators of unknown parameters $\alpha$, $\beta$ and $\gamma$ for different sample sizes have been computed. The procedure was repeated for 5,000 times and then the average estimates and standard errors (SEs) were computed. Table 1 provides the simulation results including the average estimates of the parameters and their SEs.

The results show that the MLEs is consistent as the estimated values are very close to the true values. Further, the standard errors of the estimates decrease when the sample size increases.

## Applications

Real data applications of the TLMEx distribution are discussed in this section. We have used three datasets from the field of ecology, botany, and archaeology. These datasets are listed in Appendix A. The TLMEx distribution is compared with some existing distributions which are listed in Table 2. The unknown parameters of each distribution are estimated using the maximum likelihood method. We use $-2\ell$ (where $\ell$ is the maximized log-likelihood) and AIC (Akaike information criterion) as goodness-of-fit measures.

The first dataset refers to the level of mercury in 34 albacore caught in the Eastern Mediterranean obtained from Mol *et al*. (2012). The second dataset represents the petal length (cm) for a random sample of 35 iris virginica from (Anderson, E., Bull. Amer. Iris Soc). The third dataset represents the length (cm) of a random sample

**Table 2:** Fitted distributions and their abbreviations

| Distributions | Abbreviation | Reference |
|---|---|---|
| Weibull | W | Fisher & Tippett (1928) |
| Logistic | L | Balakrishnan (1991) |
| Log-logistic | LL | de Sanatana *et al*. (2012) |
| Exponential | Ex | Akdam *et al*. (2017) |
| Chen | C | Chen (2000) |
| Lindely | Lin | Zeghdoudi & Nedjar (2016) |
| Rayleigh | R | Sarhan & Kundu (2009) |
| Muth | M | Jodra *et al*. (2015) |

**Table 3:** Estimated parameters, $-2\ell$ and AIC for fitted distributions for dataset I

| Models | Parameters | | | $-2\ell$ | *AIC* |
|---|---|---|---|---|---|
| | $\alpha$ | $\beta$ | $\gamma$ | | |
| TLMEx | 0.0899 | 1.4794 | 101.595 | 148.484 | 156.626 |
| W | 0.3091 | 3.7313 | | 152.626 | 158.446 |
| L | 2.8854 | 0.4900 | | 154.446 | 156.583 |
| LL | 2.8202 | 5.7829 | | 152.583 | 254.788 |
| Ex | 0.3423 | | | 252.788 | 243.453 |
| Lin | 0.4909 | | | 241.453 | 168.482 |
| C | 0.03541 | 1.0136 | | 164.482 | 357.911 |
| R | 0.0349 | | | 355.911 | 244.308 |
| M | 0.160.1 | | | 242.308 | 156.626 |

**Table 4:** Estimated parameters, $-2\ell$ and AIC for fitted distributions for dataset II

| Models | Parameters | | | $-2\ell$ | *AIC* |
|---|---|---|---|---|---|
| | $\alpha$ | $\beta$ | $\gamma$ | | |
| TLMEx | 34.0425 | 1.36083 | 28.3063 | 53.737 | 59.737 |
| W | 0.174431 | 9.78077 | | 63.6086 | 67.6086 |
| L | 5.43979 | 0.310664 | | 57.3134 | 61.3134 |
| LL | 5.42663 | 17.7535 | | 56.1235 | 60.1235 |
| Ex | 0.3423 | | | 189.041 | 191.041 |
| Lin | 0.3423 | | | 176.813 | 178.813 |
| C | 0.114957 | 0.552966 | | 163.773 | 167.773 |
| R | $5.8\times10^{-13}$ | | | 383.399 | 385.399 |
| M | 0.0478615 | | | 166.883 | 168.883 |

**Table 5:** Estimated parameters, $-2\ell$ and AIC for fitted distributions for dataset III

| Models | Parameters | | | $-2\ell$ | *AIC* |
|---|---|---|---|---|---|
| | $\alpha$ | $\beta$ | $\gamma$ | | |
| TLMEx | 0.1062 | 2.7769 | 54.8515 | 38.6657 | 44.6657 |
| W | 0.5345 | 27.6523 | | 914.712 | 918.712 |
| L | 1.3449 | 0.2579 | | 41.9742 | 45.9742 |
| LL | 1.3042 | 5.0169 | | 41.7554 | 45.7554 |
| Ex | 0.7362 | | | 88.8162 | 90.8162 |
| Lin | 0.9254 | | | 86.5293 | 88.5293 |
| C | 0.1629 | 1.5036 | | 43.103 | 47.103 |
| R | 0.8294 | | | 58.5767 | 60.5767 |
| M | 0.7307 | | | 87.6633 | 89.6633 |

of 61 projectile points found at the Wind Mountain Archaeological by Woosley and McIntyre (1996).

The estimated parameters of the fitted models, $-2\ell$ and AIC are given in Tables 3–5 for the three datasets, respectively. The values in these tables reveal that the TLMEx distribution provides better fit as compared with the other models. The fitted density, empirical CDF and total time on test (TTT) plots of the TLMEx distribution are displayed in Figures 3 to 5 for the three datasets, respectively. These figures show that the TLMEx distribution fits to all datasets properly. The TTT plot shows that all datasets have monotonic increasing of HRFs.

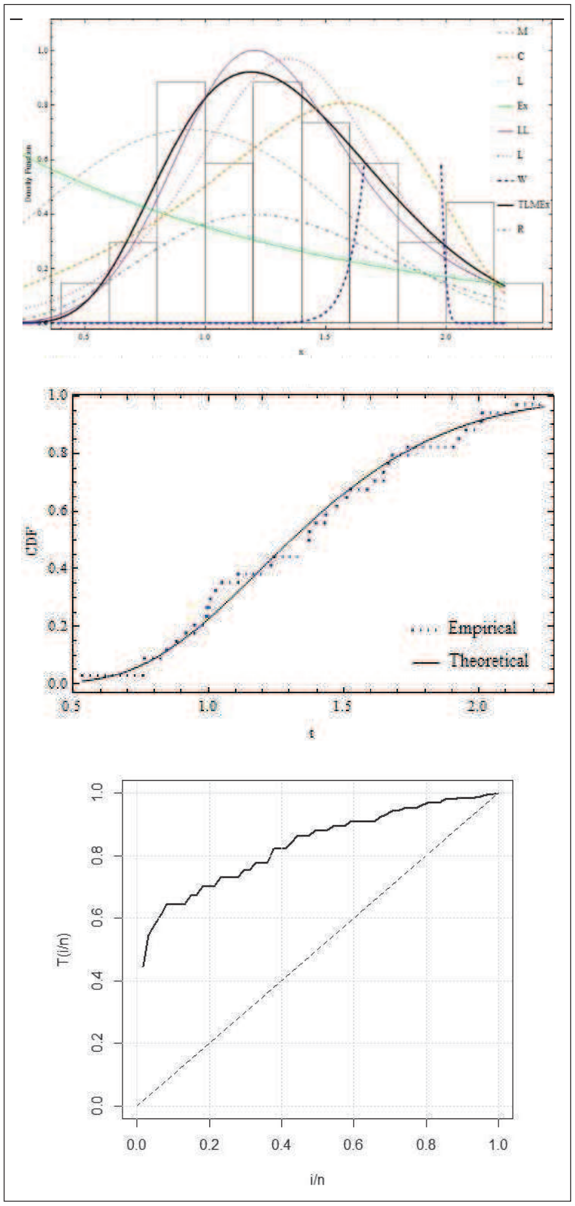


**Figure 3:** Fitted density, empirical CDF and TTT plots of the TLMEx distribution for dataset I

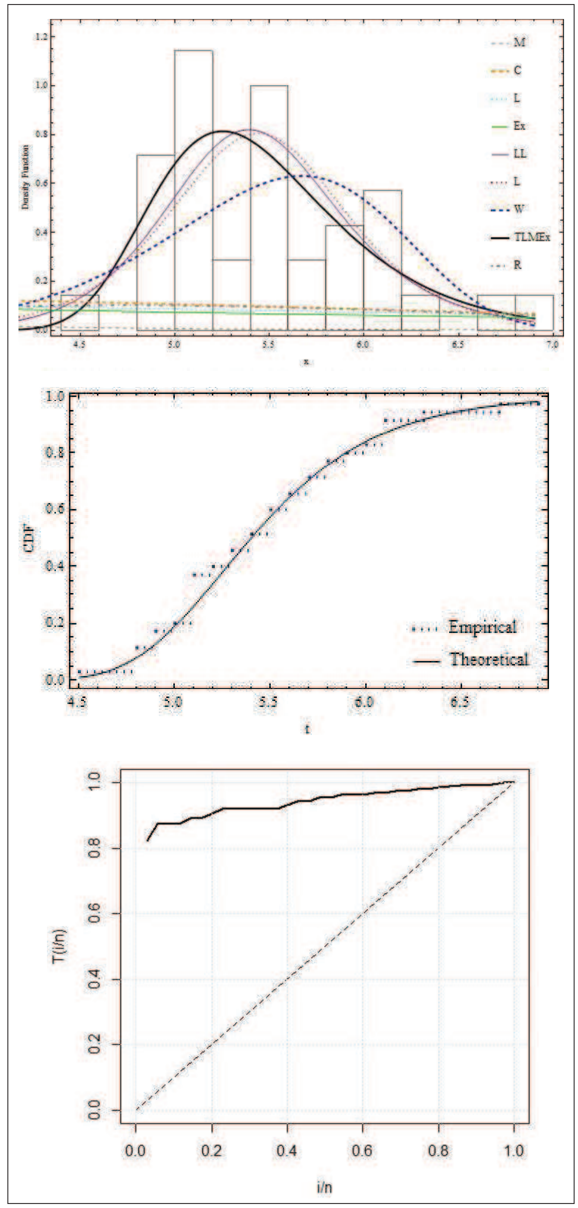


**Figure 4:** Fitted density, empirical CDF and TTT plots of the TLMEx distribution for dataset II

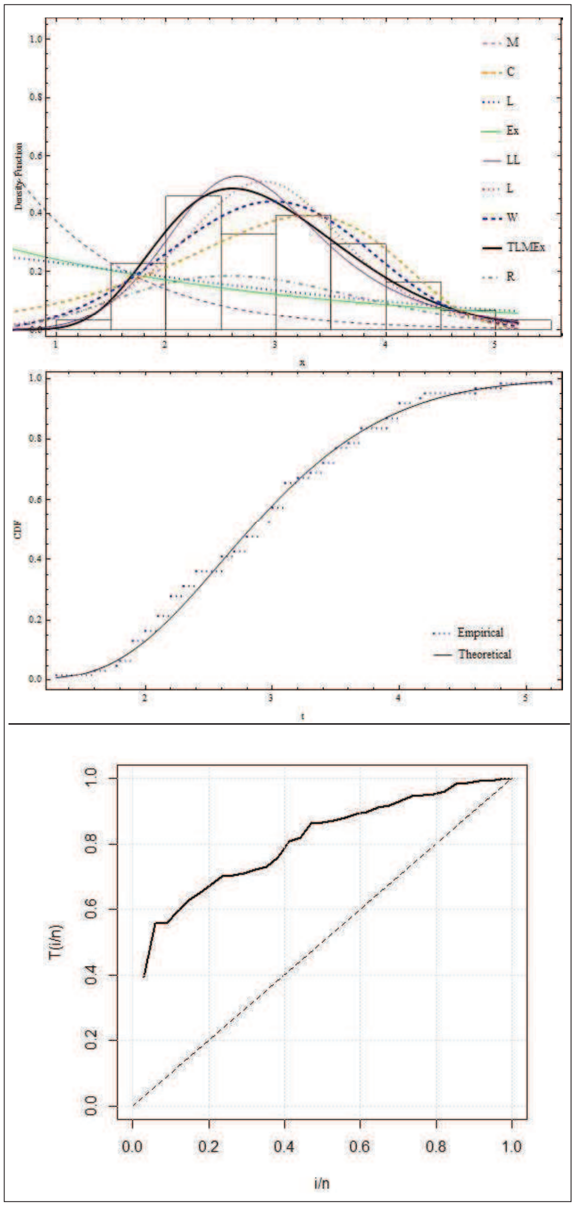


**Figure 5:** Fitted density, empirical CDF and TTT plots of the TLMEx distribution for dataset III

From Tables 3 to 5 we can see that the TLMEx distribution provides best fit to all three datasets as the AIC value for this distribution is the smallest for all three datasets. The same is reflected from Figures 3 to 5.

## CONCLUSIONS

In this paper we have introduced a new three-parameter model called Topp-Leone moment exponential (TLMEx) distribution, which generalises the moment exponential distribution proposed by Dara and Ahmad (2012). The new TLMEx model is found to be more flexible and adaptable for modelling life time data in ecology, reliability, and environmental sciences. Some mathematical properties of the proposed model are studied. A suitable characterisation of the TLMEx distribution is presented. The unknown parameters of the TLMEx model are estimated via the maximum likelihood method and evaluated through a simulation study. Applicability of the new distribution is illustrated by means of three real datasets from the field of ecology, Botany, and Archaeology. The proposed model is a reasonably better fit to the three datasets used as compared with the competing models.

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**Appendix A**

**Fisher information matrix entries**

$$J_{\alpha\alpha} = -\frac{n}{\alpha^2}, \qquad J_{\alpha\beta} = \gamma \sum_{i=1}^{n} \frac{\beta K_i}{S_i} + \sum_{i=1}^{n} -\frac{\gamma\beta K_i S_i{}^{\gamma-1}}{2-S_i{}^{\gamma}},$$

$$J_{\gamma\gamma} = -\frac{n}{\gamma^2} - \sum_{i=1}^{n} \frac{S_i{}^{2\gamma}\log(S_i)^2}{(1-S_i{}^{\gamma})^2} - \sum_{i=1}^{n} \frac{S_i{}^{\gamma}\log(S_i)^2}{1-S_i{}^{\gamma}} + (1-\alpha)\sum_{i=1}^{n} \frac{S_i{}^{2\gamma}\log(S_i)^2}{(2-S_i{}^{\gamma})^2} + (1-\alpha)\sum_{i=1}^{n} \frac{S_i{}^{\gamma}\log(S_i)^2}{2-S_i{}^{\gamma}},$$

$$J_{\alpha\gamma} = \sum_{i=1}^{n} \log(S_i),$$

$$J_{\beta\gamma} = \alpha \sum_{i=1}^{n} \frac{\beta K_i}{S_i} - \sum_{i=1}^{n} \frac{\gamma\beta K_i S_i{}^{2\gamma-1}\log(S_i)}{(1-S_i{}^{\gamma})^2} - \sum_{i=1}^{n} \frac{\beta K_i S_i{}^{\gamma-1}}{1-S_i{}^{\gamma}} - \sum_{i=1}^{n} \frac{\gamma\beta K_i S_i{}^{\gamma-1}\log(S_i)}{1-S_i{}^{\gamma}}$$
$$+ (1-\alpha)\sum_{i=1}^{n} \frac{\gamma\beta K_i S_i{}^{2\gamma-1}\log(S_i)}{(2-S_i{}^{\gamma})^2} + (1-\alpha)\sum_{i=1}^{n} \frac{\beta K_i S_i{}^{\gamma-1}}{2-S_i{}^{\gamma}}$$
$$+ (1-\alpha)\sum_{i=1}^{n} \frac{\gamma\beta K_i S_i{}^{\gamma-1}\log(S_i)}{2-S_i{}^{\gamma}}$$

and

$$J_{\beta\beta} = 2\sum_{i=1}^{n} \log\left(x_i e^{-\beta x_i}\right) + 4\beta \sum_{i=1}^{n} -x_i - (\alpha\gamma - 1)\sum_{i=1}^{n} \frac{(\beta K_i)^2}{S_i{}^2} + (\alpha\gamma - 1)\sum_{i=1}^{n} \frac{K_i(1-\beta x_i)}{S_i}$$
$$- \sum_{i=1}^{n} \frac{(\gamma\beta K_i)^2 S_i{}^{2\gamma-2}}{(1-S_i{}^{\gamma})^2} - \sum_{i=1}^{n} \frac{(-1+\gamma)\gamma(\beta K_i)^2 S_i{}^{\gamma-2}}{1-S_i{}^{\gamma}} - \sum_{i=1}^{n} \frac{\gamma K_i(1-\beta x_i) S_i{}^{\gamma-1}}{1-S_i{}^{\gamma}}$$
$$+ (1-\alpha)\sum_{i=1}^{n} \frac{(\gamma\beta K_i)^2 S_i{}^{2\gamma-2}}{(2-S_i{}^{\gamma})^2} + (1-\alpha)\sum_{i=1}^{n} \frac{\gamma(\gamma-1)(\beta K_i)^2 S_i{}^{\gamma-2}}{2-S_i{}^{\gamma}}$$
$$+ (1-\alpha)\sum_{i=1}^{n} \frac{\gamma K_i(1-\beta x_i) S_i{}^{\gamma-1}}{2-S_i{}^{\gamma}},$$

where $S_i = 1-(1+\beta x_i)e^{-\beta x_i}$ and $K_i = x_i^2 e^{-\beta x_i}$.

**The three datasets:**

Dataset I:
1.007,1.447,0.763,2.010,1.346,1.243,1.586,0.821,1.735,1.396,1.109,0.993,2.007,1.373,2.242,1.647,1.350,0.948,1.501,1.907,1.952,0.996,1.433,0.866,1.049,1.665,2.139,0.534,1.027,1.678,1.214,0.905,1.525,0.763.

Dataset II:
5.099999905, 5.800000191, 6.300000191, 6.099999905, 5.099999905, 5.5, 5.300000191, 5.5,6.900000095, 5,4.900000095, 6, 4.800000191, 6.099999905, 5.599999905, 5.099999905, 5.599999905, 4.800000191, 5.400000095, 5.099999905, 5.099999905, 5.900000095, 5.199999809, 5.699999809, 5.400000095, 4.5, 6.099999905, 5.300000191, 5.5, 6.699999809, 5.699999809, 4.900000095, 4.800000191, 5.800000191, 5.099999905.

Dataset III:
3.1, 4.1, 1.8, 2.1, 2.2, 1.3, 1.7, 3, 3.7, 2.3, 2.6, 2.2, 2.8, 3, 3.2, 3.3, 2.4, 2.8, 2.8, 2.9, 2.9, 2.2, 2.4, 2.1, 3.4, 3.1, 1.6, 3.1, 3.5, 2.3, 3.1, 2.7, 2.1, 2, 4.8, 1.9, 3.9, 2, 5.2, 2.2, 2.6, 1.9, 4, 3, 3.4, 4.2, 2.4, 3.5, 3.1, 3.7, 3.7, 2.9, 2.6, 3.6, 3.9, 3.5, 1.9, 4, 4, 4.6, 1.9.