

## RESEARCH ARTICLE

# Bayesian estimation of 3-component mixture of Gumbel type-II distributions under non-informative and informative priors

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**Abstract:** This paper deals with 3-component mixture of the Gumbel type-II distributions when the scale parameter is known under Bayesian view point. The type-I right censored sampling scheme is considered due to its extensive use in reliability theory and survival analysis, taking different non-informative and informative priors. Bayes estimates of the parameters of the mixture model along with their posterior risks are derived under different loss functions. In case where no or little prior information is available, elicitation of hyperparameters is given. In order to numerically study the execution of the Bayes estimators under different loss functions, their statistical properties have been simulated for different sample sizes and test termination times. The comparisons among the estimators have been made in terms of the corresponding posterior risks. A real life data example is also given to illustrate the study.

**Keywords:** Bayes estimators, censoring, loss functions, mixture models, posterior risks.

## INTRODUCTION

The Gumbel distribution was introduced by the German mathematician Emil Gumbel in 1958, and is useful in predicting the chance of meteorological phenomena, such as annual flood flows, earthquakes and other natural disasters. It has also been used in describing the life expectancy of components. In hydrology, the Gumbel distribution is used to analyse the variables such as monthly, quarterly and annual maximum values of daily rainfall, wind speed and river discharge volumes. The Gumbel distribution is a special case of the generalised extreme value distribution (also identified as the Fisher-Tippett distribution), and the distribution is also known as the log-Weibull distribution and the double exponential distribution (also considered as the Laplace distribution).

Many authors have discussed the Gumbel type-II distribution. For example, Kotz and Nadarajah (2000) investigated some properties of the Gumbel distribution. Feroze and Aslam (2012) applied the Bayesian estimation scheme for Gumbel type-II distribution under doubly censored samples by considering various loss functions. Furthermore, Salinas *et al.* (2012) proposed goodness-of-fit tests for the Gumbel distribution with type-II right censored data. Abbas *et al.* (2013) obtained the Bayes estimators for the parameters of the Gumbel type-II distribution under different loss functions and compared these estimates with similar performance by the maximum likelihood method. Reyad and Ahmed (2015) proposed the E-Bayesian analysis of the unknown shape parameter of the Gumbel type-II distribution based on type-II censored samples. These estimators are obtained under different symmetric and asymmetric loss functions.

Mixture models play an important role in many applicable fields such as medicine, psychology, cluster analysis, life testing and reliability analysis. A finite mixture of some suitable probability distributions is recommended to study a population that is supposed to comprise a number of subpopulations mixing in unknown proportion. However, several researchers are interested in different parameters of mixture distributions. Chen *et al.* (1989) considered the Bayes estimation for mixtures of two Weibull distributions under type-I censoring. Saleem *et al.* (2010) presented the Bayesian analysis of the 2-component mixture of power function distributions using complete and censored data. Kazmi *et al.* (2012) developed the Bayesian analysis of 2-component mixture of the Maxwell distributions, while Feroze

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and Aslam (2013) discussed the Bayesian estimation of 2-component mixture of Gumbel type-II distribution under informative priors using different loss functions. Noor and Aslam (2013) studied Bayesian inference of the inverse Weibull mixture model under type-I censoring, and Ali (2015) described mixture of inverse Rayleigh distribution under Bayesian framework.

One particular feature, often present in time-to-event data, is known as censoring. Which broadly speaking, occurs when some lifetimes are known to have occurred only within certain intervals. The remainders of the lifetimes are known exactly. There are various categories of censoring such as right censoring, left censoring and interval censoring, single or multiple censoring, and type-I and type-II censoring. Type-I and type-II censoring schemes are most familiar among them.

Motivated by the above mentioned applications of mixture models, the authors have planned to conduct Bayesian analysis of a 3-component mixture of the Gumbel type-II distributions with unknown mixing proportions. The parameters of component distributions are assumed to be unknown. Three different priors and three different loss functions are used for the Bayesian analysis. In addition, an ordinary type-I right censored sampling scheme is also assumed.

**METHODOLOGY**

**3-component mixture of Gumbel type-II distributions**

The probability density function (PDF) and the cumulative distribution function (CDF) of the Gumbel type-II distribution for a random variable  $x$  are given by:

$$f_m(x; \alpha_m, \beta_m) = \alpha_m \beta_m x^{-(\alpha_m+1)} \exp(-\beta_m x^{-\alpha_m}), x \geq 0, \alpha_m > 0, \beta_m > 0, m = 1, 2, 3 \quad \dots(1)$$

$$F_m(x) = 1 - \exp(-\beta_m x^{-\alpha_m}), m = 1, 2, 3 \quad \dots(2)$$

Where  $\alpha_m$  and  $\beta_m$  are the parameters of Gumbel type-II distribution.

A finite 3-component mixture model with the unknown mixing proportions  $p_1$  and  $p_2$  is defined as when the scale parameter  $\alpha_m = 1$

$$f(x) = p_1 f_1(x) + p_2 f_2(x) + (1 - p_1 - p_2) f_3(x) \quad \dots(3)$$

$$p_1, p_2 \geq 0, p_1 + p_2 \leq 1$$

$$f(x; \beta_1, \beta_2, \beta_3, p_1, p_2) = p_1 \beta_1 x^{-2} \exp(-\beta_1 x^{-1}) + p_2 \beta_2 x^{-2} \exp(-\beta_2 x^{-1}) + (1 - p_1 - p_2) \beta_3 x^{-2} \exp(-\beta_3 x^{-1}) \quad \dots(4)$$

While the CDF of 3-component mixture of the Gumbel type-II distributions is given by:

$$F(x) = p_1 F_1(x) + p_2 F_2(x) + (1 - p_1 - p_2) F_3(x) \quad \dots(5)$$

$$F(x) = 1 - p_1 \exp(-\beta_1 x^{-1}) - p_2 \exp(-\beta_2 x^{-1}) - (1 - p_1 - p_2) \exp(-\beta_3 x^{-1}) \quad \dots(6)$$

**The likelihood function**

Suppose ‘n’ units from the 3-component mixture of Gumbel type-II distributions are used in a life testing experiment with fixed test termination time ‘t’. Let ‘r’ units out of ‘n’ units failed until fixed test termination time ‘t’ and the remaining (n-r) units are still working. According to Mendenhall and Hader (1958), there are many practical situations in which the failed objects can be pointed out easily as a subset of subpopulation-I, subpopulation-II or subpopulation-III. Out of ‘r’ units, suppose  $r_1, r_2$  and  $r_3$  units belong to subpopulation-I, subpopulation-II or subpopulation-III, respectively and such that  $r = r_1 + r_2 + r_3$ . Now we define  $x_{lk}, 0 < x_{lk} < t$  be the failure time of  $k^{th}$  unit belonging to the  $l^{th}$  subpopulation, where  $l = 1, 2, 3$  and  $k = 1, 2, \dots, r_l$ . For a 3-component mixture model, the likelihood function can be written as

$$L(\psi | x) \propto \left\{ \prod_{k=1}^{r_1} p_1 f_1(x_{1k}) \right\} \left\{ \prod_{k=1}^{r_2} p_2 f_2(x_{2k}) \right\} \left\{ \prod_{k=1}^{r_3} (1 - p_1 - p_2) f_3(x_{3k}) \right\} [1 - F(t)]^{n-r} \quad \dots(7)$$

After simplification, the likelihood function of 3-component mixture of Gumbel type-II distributions is given by:

$$L(\psi | x) \propto \beta_1^{r_1} \beta_2^{r_2} \beta_3^{r_3} \sum_{i=0}^{n-r} \sum_{j=0}^i \binom{n-r}{i} \binom{i}{j} \exp\left\{-\beta_1 \left( (n-r-i)t^{-1} + \sum_{k=1}^{r_1} x_{1k}^{-1} \right)\right\} \exp\left\{-\beta_2 \left( (i-j)t^{-1} + \sum_{k=1}^{r_2} x_{2k}^{-1} \right)\right\} \exp\left\{-\beta_3 \left( jt^{-1} + \sum_{k=1}^{r_3} x_{3k}^{-1} \right)\right\} \cdot p_1^{n-r-i+r_1} p_2^{i-j+r_2} (1 - p_1 - p_2)^{j+r_3} \quad \dots(8)$$

Where  $X = (x_{11}, x_{12}, \dots, x_{1r_1}, x_{21}, x_{22}, \dots, x_{2r_2}, x_{31}, x_{32}, \dots, x_{3r_3})$  are the observed failure times for the uncensored observations and  $\psi = (\beta_1, \beta_2, \beta_3, p_1, p_2)$ .

**Posterior distribution using the non-informative and the informative priors (IP)**

In this section, posterior distributions of parameters given data, say  $x$ , are derived using the non-informative (uniform and Jeffreys) and the informative (gamma) priors.

**Posterior distribution using uniform prior (UP)**

When the elicitation of hyperparameters is difficult or little prior information is given, usually the non-informative prior is assumed to be the UP. UPs over the intervals  $(0, \infty)$  and  $(0, 1)$  are taken for the parameters  $\beta_1, \beta_2$  and  $\beta_3$  of Gumbel type-II distribution and for the mixing proportions  $(p_1, p_2)$ , respectively. With these settings, joint prior distribution of parameters  $(\beta_1, \beta_2, \beta_3, p_1, p_2)$ , as defined by Saleem *et al.* (2010), is given by:

$$\pi_1(\psi) \propto 1; \beta_1, \beta_2, \beta_3 > 0, p_1, p_2 \geq 0, p_1 + p_2 \leq 1 \quad \dots(9)$$

The joint posterior distribution of parameters  $\beta_1, \beta_2, \beta_3, p_1$  and  $p_2$  given data  $x$  assuming the UP is:

$$g_1(\psi|x) = \frac{L(\psi|x)\pi_1(\psi)}{\int_{\psi} L(\psi|x)\pi_1(\psi)d\psi} \quad \dots(10)$$

By using equations (8) and (9) in equation (10), we can obtain the following:

$$g_1(\psi|x) = Q_1^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^i \binom{n-r}{i} \binom{i}{j} \beta_1^{A_{11}-1} \beta_2^{A_{21}-1} \beta_3^{A_{31}-1} \exp(-\beta_1 B_{11}) \exp(-\beta_2 B_{21}) \exp(-\beta_3 B_{31}) P_1^{A_{01}-1} P_2^{B_{01}-1} (1-p_1-p_2)^{C_{01}-1} \quad \dots(11)$$

$$A_{11} = r_1 + 1, A_{21} = r_2 + 1, A_{31} = r_3 + 1, B_{11} = \sum_{k=1}^{r_1} x_{1k}^{-1} + (n-r-i)t^{-1},$$

$$B_{21} = \sum_{k=1}^{r_2} x_{2k}^{-1} + (i-j)t^{-1}, B_{31} = \sum_{k=1}^{r_3} x_{3k}^{-1} + jt^{-1},$$

$$A_{01} = n-r-i+r_1+1, B_{01} = i-j+r_2+1, C_{01} = j+r_3+1,$$

$$Q_1 = \sum_{i=0}^{n-r} \sum_{j=0}^i \binom{n-r}{i} \binom{i}{j} B(A_{01}, C_{01}) B(B_{01}, A_{01} + C_{01}) \frac{\Gamma(A_{11}) \Gamma(A_{21}) \Gamma(A_{31})}{B_{11}^{A_{11}} B_{21}^{A_{21}} B_{31}^{A_{31}}}$$

**Posterior distribution using the Jeffreys prior (JP)**

According to Jeffreys (1946; 1961) and Berger (1985), the JP is defined as, where  $p(\beta_m) \propto \sqrt{|I(\beta_m)|}$ , where  $I(\beta_m) = -E\left[\frac{\partial^2 f(x, \beta_m)}{\partial \beta_m^2}\right]$  is the Fisher's information matrix. The prior distributions of the mixing proportions  $p_1$  and  $p_2$  are again taken to be uniform over the interval  $(0, 1)$ . The joint prior distribution of parameters  $\beta_1, \beta_2, \beta_3, p_1$  and  $p_2$  is:

$$\pi_2(\psi) \propto \frac{1}{\beta_1 \beta_2 \beta_3}, \beta_1, \beta_2, \beta_3 \geq 0, p_1, p_2 \geq 0, p_1 + p_2 \leq 1, \quad \dots(12)$$

The joint posterior distribution of parameters  $\beta_1, \beta_2, \beta_3, p_1$  and  $p_2$  given data  $x$  assuming the JP is:

$$g_2(\psi|x) = \frac{L(\psi|x)\pi_2(\psi)}{\int_{\psi} L(\psi|x)\pi_2(\psi)d\psi} \quad \dots(13)$$

By using equations (8) and (12) in equation (13), we can obtain the following:

$$g_2(\psi|x) = Q_2^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^i \binom{n-r}{i} \binom{i}{j} \beta_1^{A_{12}-1} \beta_2^{A_{22}-1} \beta_3^{A_{32}-1} \exp(-\beta_1 B_{12}) \exp(-\beta_2 B_{22}) \exp(-\beta_3 B_{32}) P_1^{A_{02}-1} P_2^{B_{02}-1} (1-p_1-p_2)^{C_{02}-1} \quad \dots(14)$$

where

$$A_{12} = r_1, A_{22} = r_2, A_{32} = r_3, B_{12} = \sum_{k=1}^{r_1} x_{1k}^{-1} + (n-r-i)t^{-1},$$

$$B_{22} = \sum_{k=1}^{r_2} x_{2k}^{-1} + (i-j)t^{-1},$$

$$B_{32} = \sum_{k=1}^{r_3} x_{3k}^{-1} + jt^{-1}, A_{02} = n-r-i+r_1+1, B_{02} = i-j+r_2+1,$$

$$C_{02} = j+r_3+1,$$

$$Q_2 = \sum_{i=0}^{n-r} \sum_{j=0}^i \binom{n-r}{i} \binom{i}{j} B(A_{02}, C_{02}) B(B_{02}, A_{02} + C_{02}) \frac{\Gamma(A_{12}) \Gamma(A_{22}) \Gamma(A_{32})}{B_{12}^{A_{12}} B_{22}^{A_{22}} B_{32}^{A_{32}}}$$

**Posterior distribution using the gamma prior (GP)**

As an informative prior, we take the gamma prior for the component parameters  $\beta_1, \beta_2, \beta_3$  and bivariate beta prior for proportion parameters  $p_1, p_2$ . Symbolically, it can be written as:  $\beta_1 \sim \text{Gamma}(a_1, b_1), \beta_2 \sim \text{Gamma}(a_2, b_2), \beta_3 \sim \text{Gamma}(a_3, b_3)$  and  $p_1, p_2 \sim$  bivariate beta  $(a, b, c)$ . Again assuming

independence of all parameters, the joint prior distribution of  $(\beta_1, \beta_2, \beta_3, p_1, p_2)$  is given by:

$$\pi_3(\phi) \propto \beta_1^{a_1-1} \exp(-b_1\beta_1) \beta_2^{a_2-1} \exp(-b_2\beta_2) \beta_3^{a_3-1} \exp(-b_3\beta_3) p_1^{a_1-1} p_2^{b-1} (1-p_1-p_2)^{c-1} \dots(15)$$

The joint posterior distribution of parameters  $\beta_1, \beta_2, \beta_3, p_1$  and  $p_2$  given data  $x$

$$g_3(\psi|x) = \frac{L(\psi|x)\pi_3(\psi)}{\int_{\psi} L(\psi|x)\pi_3(\psi)d\psi} \dots(16)$$

By using equations (8) and (15) in equation (16), we can obtain the following:

$$g_3(\psi|x) = Q_3^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^i \binom{n-r}{j} \beta_1^{A_{13}-1} \beta_2^{A_{23}-1} \beta_3^{A_{33}-1} \exp(-\beta_1 B_{13}) \exp(-\beta_2 B_{23}) \exp(-\beta_3 B_{33}) p_1^{A_{03}-1} p_2^{B_{03}-1} (1-p_1-p_2)^{C_{03}-1} \dots(17)$$

where

$$A_{13} = r_1 + a_1, A_{23} = r_2 + a_2, A_{33} = r_3 + a_3, B_{13} = \sum_{k=1}^{r_1} x_{1k}^{-1} + (n-r-i)t^{-1} + b_1, B_{23} = \sum_{k=1}^{r_2} x_{2k}^{-1} + (i-j)t^{-1} + b_2, B_{33} = \sum_{k=1}^{r_3} x_{3k}^{-1} + jt^{-1} + b_3, A_{03} = n-r-i+r_1+a, B_{03} = i-j+r_2+b, C_{03} = j+r_3+c,$$

$$Q_3 = \sum_{i=0}^{n-r} \sum_{j=0}^i \binom{n-r}{j} B(A_{03}, C_{03}) B(B_{03}, A_{03} + C_{03}) \frac{\Gamma(A_{13}) \Gamma(A_{23}) \Gamma(A_{33})}{B_{13}^{A_{13}} B_{23}^{A_{23}} B_{33}^{A_{33}}}$$

**Bayes estimators and posterior risks using the UP, JP and IP under different loss functions**

If  $L(\beta, d)$  is a loss function, then the expected value of the loss function for a given decision with respect to the posterior distribution is posterior risk function, and if  $\hat{d}$  is a Bayes estimator, then  $\rho(\hat{d})$  is called posterior risk and is given by  $\rho(\hat{d}) = E_{\beta|x} \{L(\beta, d)\}$ .

**Bayes estimators and posterior risks using the UP, JP and IP under SELF**

The squared error loss function (SELF) has been suggested by Legendre (1805), and is defined as:  $L(\beta, d) = (\beta - d)^2$ . The Bayes estimator and posterior risk under SELF are:  $\hat{d} = E_{\beta|x}(\beta)$  and  $\rho(\hat{d}) = E_{\beta|x}(\beta^2) - \{E_{\beta|x}(\beta)\}^2$ , respectively.

So, the Bayes estimators and posterior risks using the UP, JP and IP for parameters  $\beta_1, \beta_2, \beta_3, p_1$  and  $p_2$  under SELF obtained with their respective marginal posterior distributions are given below:

$$\hat{\beta}_{1\tau} = Q_{\tau}^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^i \binom{n-r}{j} \frac{\Gamma(A_{1\tau}+1) \Gamma(A_{2\tau}) \Gamma(A_{3\tau})}{B_{1\tau}^{A_{1\tau}+1} B_{2\tau}^{A_{2\tau}} B_{3\tau}^{A_{3\tau}}} B(A_{0\tau}, C_{0\tau}) B(B_{0\tau}, A_{0\tau} + C_{0\tau}) \dots(18)$$

$$\hat{\beta}_{2\tau} = Q_{\tau}^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^i \binom{n-r}{j} \frac{\Gamma(A_{1\tau}) \Gamma(A_{2\tau}+1) \Gamma(A_{3\tau})}{B_{1\tau}^{A_{1\tau}} B_{2\tau}^{A_{2\tau}+1} B_{3\tau}^{A_{3\tau}}} B(A_{0\tau}, C_{0\tau}) B(B_{0\tau}, A_{0\tau} + C_{0\tau}) \dots(19)$$

$$\hat{\beta}_{3\tau} = Q_{\tau}^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^i \binom{n-r}{j} \frac{\Gamma(A_{1\tau}) \Gamma(A_{2\tau}) \Gamma(A_{3\tau}+1)}{B_{1\tau}^{A_{1\tau}} B_{2\tau}^{A_{2\tau}} B_{3\tau}^{A_{3\tau}+1}} B(A_{0\tau}, C_{0\tau}) B(B_{0\tau}, A_{0\tau} + C_{0\tau}) \dots(20)$$

$$\hat{p}_{1\tau} = Q_{\tau}^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^i \binom{n-r}{j} \frac{\Gamma(A_{1\tau}) \Gamma(A_{2\tau}) \Gamma(A_{3\tau})}{B_{1\tau}^{A_{1\tau}} B_{2\tau}^{A_{2\tau}} B_{3\tau}^{A_{3\tau}}} B(B_{0\tau}, C_{0\tau}) B(A_{0\tau} + 1, B_{0\tau} + C_{0\tau}) \dots(21)$$

$$\hat{p}_{2\tau} = Q_{\tau}^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^i \binom{n-r}{j} \frac{\Gamma(A_{1\tau}) \Gamma(A_{2\tau}) \Gamma(A_{3\tau})}{B_{1\tau}^{A_{1\tau}} B_{2\tau}^{A_{2\tau}} B_{3\tau}^{A_{3\tau}}} B(A_{0\tau}, C_{0\tau}) B(B_{0\tau} + 1, A_{0\tau} + C_{0\tau}) \dots(22)$$

$$\rho(\hat{\beta}_{1\tau}) = Q_{\tau}^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^i \binom{n-r}{j} \frac{\Gamma(A_{1\tau}+2) \Gamma(A_{2\tau}) \Gamma(A_{3\tau})}{B_{1\tau}^{A_{1\tau}+2} B_{2\tau}^{A_{2\tau}} B_{3\tau}^{A_{3\tau}}} B(A_{0\tau}, C_{0\tau}) B(B_{0\tau}, A_{0\tau} + C_{0\tau}) - (\hat{\beta}_{1\tau})^2 \dots(23)$$

$$\rho(\hat{\beta}_{2\tau}) = Q_{\tau}^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^i \binom{n-r}{j} \frac{\Gamma(A_{1\tau}) \Gamma(A_{2\tau}+2) \Gamma(A_{3\tau})}{B_{1\tau}^{A_{1\tau}} B_{2\tau}^{A_{2\tau}+2} B_{3\tau}^{A_{3\tau}}} B(A_{0\tau}, C_{0\tau}) B(B_{0\tau}, A_{0\tau} + C_{0\tau}) - (\hat{\beta}_{2\tau})^2 \dots(24)$$

$$\rho(\hat{\beta}_{3\tau}) = Q_{\tau}^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^i \binom{n-r}{j} \frac{\Gamma(A_{1\tau}) \Gamma(A_{2\tau}) \Gamma(A_{3\tau}+2)}{B_{1\tau}^{A_{1\tau}} B_{2\tau}^{A_{2\tau}} B_{3\tau}^{A_{3\tau}+2}} B(A_{0\tau}, C_{0\tau}) B(B_{0\tau}, A_{0\tau} + C_{0\tau}) - (\hat{\beta}_{3\tau})^2 \dots(25)$$

$$\rho(\hat{p}_{1\tau}) = Q_{\tau}^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^i \binom{n-r}{j} \frac{\Gamma(A_{1\tau}) \Gamma(A_{2\tau}) \Gamma(A_{3\tau})}{B_{1\tau}^{A_{1\tau}} B_{2\tau}^{A_{2\tau}} B_{3\tau}^{A_{3\tau}}} B(B_{0\tau}, C_{0\tau}) B(A_{0\tau} + 2, B_{0\tau} + C_{0\tau}) - (\hat{p}_{1\tau})^2 \dots(26)$$

$$\rho(\hat{p}_{2\tau}) = Q_{\tau}^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^i \binom{n-r}{j} \frac{\Gamma(A_{1\tau}) \Gamma(A_{2\tau}) \Gamma(A_{3\tau})}{B_{1\tau}^{A_{1\tau}} B_{2\tau}^{A_{2\tau}} B_{3\tau}^{A_{3\tau}}} B(A_{0\tau}, C_{0\tau}) B(B_{0\tau} + 2, A_{0\tau} + C_{0\tau}) - (\hat{p}_{2\tau})^2 \dots(27)$$

Where  $\tau = 1$  for the uniform prior,  $\tau = 2$  for the Jeffreys prior and  $\tau = 3$  for the gamma prior.

**Bayes estimators and posterior risks using the UP, JP and IP under precautionary loss function (PLF)**

Norström (1996) discussed an asymmetric precautionary loss function (PLF) and also introduced a special case of general class of PLFs, which is defined as  $L(\beta, d) = \frac{(\beta - d)^2}{d}$ .

The Bayes estimator and posterior risk are:

$$\hat{d} = \{E_{\beta|x}(\beta^2)\}^{\frac{1}{2}} \ \& \ \rho(\hat{d}) = 2\{E_{\beta|x}(\beta^2)\}^{\frac{1}{2}} - 2E_{\beta|x}(\beta), \text{ respectively.}$$

The respective marginal posterior distribution yields the Bayes estimators and posterior risk using the UP, JP and IP for parameters  $\beta_1, \beta_2, \beta_3, p_1$  and  $p_2$  under PLF as:

$$\hat{\beta}_{1r} = \left\{ Q_r^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^i \binom{n-r}{j} \frac{\Gamma(A_{1r}+2)\Gamma(A_{2r})\Gamma(A_{3r})}{B_{1r}^{A_{1r}+2} B_{2r}^{A_{2r}} B_{3r}^{A_{3r}}} B(A_{0r}, C_{0r}) B(B_{0r}, A_{0r} + C_{0r}) \right\}^{\frac{1}{2}} \dots(28)$$

$$\hat{\beta}_{2r} = \left\{ Q_r^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^i \binom{n-r}{j} \frac{\Gamma(A_{1r})\Gamma(A_{2r}+2)\Gamma(A_{3r})}{B_{1r}^{A_{1r}} B_{2r}^{A_{2r}+2} B_{3r}^{A_{3r}}} B(A_{0r}, C_{0r}) B(B_{0r}, A_{0r} + C_{0r}) \right\}^{\frac{1}{2}} \dots(29)$$

$$\hat{\beta}_{3r} = \left\{ Q_r^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^i \binom{n-r}{j} \frac{\Gamma(A_{1r})\Gamma(A_{2r})\Gamma(A_{3r}+2)}{B_{1r}^{A_{1r}} B_{2r}^{A_{2r}} B_{3r}^{A_{3r}+2}} B(A_{0r}, C_{0r}) B(B_{0r}, A_{0r} + C_{0r}) \right\}^{\frac{1}{2}} \dots(30)$$

$$\hat{p}_{1r} = \left\{ Q_r^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^i \binom{n-r}{j} \frac{\Gamma(A_{1r})\Gamma(A_{2r})\Gamma(A_{3r})}{B_{1r}^{A_{1r}} B_{2r}^{A_{2r}} B_{3r}^{A_{3r}}} B(B_{0r}, C_{0r}) B(A_{0r} + 2, B_{0r} + C_{0r}) \right\}^{\frac{1}{2}} \dots(31)$$

$$\hat{p}_{2r} = \left\{ Q_r^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^i \binom{n-r}{j} \frac{\Gamma(A_{1r})\Gamma(A_{2r})\Gamma(A_{3r})}{B_{1r}^{A_{1r}} B_{2r}^{A_{2r}} B_{3r}^{A_{3r}}} B(A_{0r}, C_{0r}) B(B_{0r} + 2, A_{0r} + C_{0r}) \right\}^{\frac{1}{2}} \dots(32)$$

$$\rho(\hat{\beta}_1) = 2 \left\{ Q_r^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^i \binom{n-r}{j} \frac{\Gamma(A_{1r}+2)\Gamma(A_{2r})\Gamma(A_{3r})}{B_{1r}^{A_{1r}+2} B_{2r}^{A_{2r}} B_{3r}^{A_{3r}}} B(A_{0r}, C_{0r}) B(B_{0r}, A_{0r} + C_{0r}) \right\}^{\frac{1}{2}} - 2 \left\{ Q_r^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^i \binom{n-r}{j} \frac{\Gamma(A_{1r}+1)\Gamma(A_{2r})\Gamma(A_{3r})}{B_{1r}^{A_{1r}+1} B_{2r}^{A_{2r}} B_{3r}^{A_{3r}}} B(A_{0r}, C_{0r}) B(B_{0r}, A_{0r} + C_{0r}) \right\} \dots(33)$$

$$\rho(\hat{\beta}_2) = 2 \left\{ Q_r^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^i \binom{n-r}{j} \frac{\Gamma(A_{1r})\Gamma(A_{2r}+2)\Gamma(A_{3r})}{B_{1r}^{A_{1r}} B_{2r}^{A_{2r}+2} B_{3r}^{A_{3r}}} B(A_{0r}, C_{0r}) B(B_{0r}, A_{0r} + C_{0r}) \right\}^{\frac{1}{2}} - 2 \left\{ Q_r^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^i \binom{n-r}{j} \frac{\Gamma(A_{1r})\Gamma(A_{2r}+1)\Gamma(A_{3r})}{B_{1r}^{A_{1r}} B_{2r}^{A_{2r}+1} B_{3r}^{A_{3r}}} B(A_{0r}, C_{0r}) B(B_{0r}, A_{0r} + C_{0r}) \right\} \dots(34)$$

$$\rho(\hat{\beta}_3) = 2 \left\{ Q_r^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^i \binom{n-r}{j} \frac{\Gamma(A_{1r})\Gamma(A_{2r})\Gamma(A_{3r}+2)}{B_{1r}^{A_{1r}} B_{2r}^{A_{2r}} B_{3r}^{A_{3r}+2}} B(A_{0r}, C_{0r}) B(B_{0r}, A_{0r} + C_{0r}) \right\}^{\frac{1}{2}} - 2 \left\{ Q_r^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^i \binom{n-r}{j} \frac{\Gamma(A_{1r})\Gamma(A_{2r})\Gamma(A_{3r}+1)}{B_{1r}^{A_{1r}} B_{2r}^{A_{2r}} B_{3r}^{A_{3r}+1}} B(A_{0r}, C_{0r}) B(B_{0r}, A_{0r} + C_{0r}) \right\} \dots(35)$$

$$\rho(\hat{p}_1) = 2 \left\{ Q_r^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^i \binom{n-r}{j} \frac{\Gamma(A_{1r})\Gamma(A_{2r})\Gamma(A_{3r})}{B_{1r}^{A_{1r}} B_{2r}^{A_{2r}} B_{3r}^{A_{3r}}} B(B_{0r}, C_{0r}) B(A_{0r} + 2, B_{0r} + C_{0r}) \right\}^{\frac{1}{2}} - 2 \left\{ Q_r^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^i \binom{n-r}{j} \frac{\Gamma(A_{1r})\Gamma(A_{2r})\Gamma(A_{3r})}{B_{1r}^{A_{1r}} B_{2r}^{A_{2r}} B_{3r}^{A_{3r}}} B(B_{0r}, C_{0r}) B(A_{0r} + 1, B_{0r} + C_{0r}) \right\} \dots(36)$$

$$\rho(\hat{p}_2) = 2 \left\{ Q_r^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^i \binom{n-r}{j} \frac{\Gamma(A_{1r})\Gamma(A_{2r})\Gamma(A_{3r})}{B_{1r}^{A_{1r}} B_{2r}^{A_{2r}} B_{3r}^{A_{3r}}} B(A_{0r}, C_{0r}) B(B_{0r} + 2, A_{0r} + C_{0r}) \right\}^{\frac{1}{2}} - 2 \left\{ Q_r^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^i \binom{n-r}{j} \frac{\Gamma(A_{1r})\Gamma(A_{2r})\Gamma(A_{3r})}{B_{1r}^{A_{1r}} B_{2r}^{A_{2r}} B_{3r}^{A_{3r}}} B(A_{0r}, C_{0r}) B(B_{0r} + 1, A_{0r} + C_{0r}) \right\} \dots(37)$$

**Bayes estimators and posterior risks using the UP, the JP and IP under DeGroot loss function (DLF)**

DeGroot (2005) introduced the asymmetric loss function,  $L(\beta, d) = \left(\frac{\beta - d}{d}\right)^2$  known as DeGroot loss function (DLF). The Bayes estimator and its posterior risk under DLF are:  $\hat{d} = \frac{E_{\beta|x}(\beta^2)}{E_{\beta|x}(\beta)}$  and  $\rho(\hat{d}) = 1 - \frac{\{E_{\beta|x}(\beta)\}^2}{E_{\beta|x}(\beta^2)}$ , respectively. The Bayes estimators and posterior risks using the UP, JP and IP for parameters  $\beta_1, \beta_2, \beta_3, p_1$  and  $p_2$  under DLF are:

$$\hat{\beta}_{1r} = \frac{\left\{ Q_r^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^i \binom{n-r}{j} \frac{\Gamma(A_{1r}+2)\Gamma(A_{2r})\Gamma(A_{3r})}{B_{1r}^{A_{1r}+2} B_{2r}^{A_{2r}} B_{3r}^{A_{3r}}} B(A_{0r}, C_{0r}) B(B_{0r}, A_{0r} + C_{0r}) \right\}}{\left\{ Q_r^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^i \binom{n-r}{j} \frac{\Gamma(A_{1r}+1)\Gamma(A_{2r})\Gamma(A_{3r})}{B_{1r}^{A_{1r}+1} B_{2r}^{A_{2r}} B_{3r}^{A_{3r}}} B(A_{0r}, C_{0r}) B(B_{0r}, A_{0r} + C_{0r}) \right\}} \dots(38)$$

$$\hat{\beta}_{2r} = \frac{\left\{ Q_r^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^i \binom{n-r}{j} \frac{\Gamma(A_{1r})\Gamma(A_{2r}+2)\Gamma(A_{3r})}{B_{1r}^{A_{1r}} B_{2r}^{A_{2r}+2} B_{3r}^{A_{3r}}} B(A_{0r}, C_{0r}) B(B_{0r}, A_{0r} + C_{0r}) \right\}}{\left\{ Q_r^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^i \binom{n-r}{j} \frac{\Gamma(A_{1r})\Gamma(A_{2r}+1)\Gamma(A_{3r})}{B_{1r}^{A_{1r}} B_{2r}^{A_{2r}+1} B_{3r}^{A_{3r}}} B(A_{0r}, C_{0r}) B(B_{0r}, A_{0r} + C_{0r}) \right\}} \dots(39)$$

$$\hat{\beta}_{3r} = \frac{\left\{ Q_r^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^i \binom{n-r}{j} \frac{\Gamma(A_{1r})\Gamma(A_{2r})\Gamma(A_{3r}+2)}{B_{1r}^{A_{1r}} B_{2r}^{A_{2r}} B_{3r}^{A_{3r}+2}} B(A_{0r}, C_{0r}) B(B_{0r}, A_{0r} + C_{0r}) \right\}}{\left\{ Q_r^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^i \binom{n-r}{j} \frac{\Gamma(A_{1r})\Gamma(A_{2r})\Gamma(A_{3r}+1)}{B_{1r}^{A_{1r}} B_{2r}^{A_{2r}} B_{3r}^{A_{3r}+1}} B(A_{0r}, C_{0r}) B(B_{0r}, A_{0r} + C_{0r}) \right\}} \dots(40)$$

$$\hat{p}_{1r} = \frac{\left\{ Q_r^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^i \binom{n-r}{i} \binom{i}{j} \frac{\Gamma(A_{1r}) \Gamma(A_{2r}) \Gamma(A_{3r})}{B_{1r}^{A_{1r}} B_{2r}^{A_{2r}} B_{3r}^{A_{3r}}} B(B_{0r}, C_{0r}) B(A_{0r} + 2, B_{0r} + C_{0r}) \right\}}{\left\{ Q_r^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^i \binom{n-r}{i} \binom{i}{j} \frac{\Gamma(A_{1r}) \Gamma(A_{2r}) \Gamma(A_{3r})}{B_{1r}^{A_{1r}} B_{2r}^{A_{2r}} B_{3r}^{A_{3r}}} B(B_{0r}, C_{0r}) B(A_{0r} + 1, B_{0r} + C_{0r}) \right\}} \quad \dots(41)$$

$$\hat{p}_{2r} = \frac{\left\{ Q_r^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^i \binom{n-r}{i} \binom{i}{j} \frac{\Gamma(A_{1r}) \Gamma(A_{2r}) \Gamma(A_{3r})}{B_{1r}^{A_{1r}} B_{2r}^{A_{2r}} B_{3r}^{A_{3r}}} B(A_{0r}, C_{0r}) B(B_{0r} + 2, A_{0r} + C_{0r}) \right\}}{\left\{ Q_r^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^i \binom{n-r}{i} \binom{i}{j} \frac{\Gamma(A_{1r}) \Gamma(A_{2r}) \Gamma(A_{3r})}{B_{1r}^{A_{1r}} B_{2r}^{A_{2r}} B_{3r}^{A_{3r}}} B(A_{0r}, C_{0r}) B(B_{0r} + 1, A_{0r} + C_{0r}) \right\}} \quad \dots(42)$$

$$\rho(\hat{p}_{1r}) = 1 - \frac{\left\{ Q_r^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^i \binom{n-r}{i} \binom{i}{j} \frac{\Gamma(A_{1r} + 1) \Gamma(A_{2r}) \Gamma(A_{3r})}{B_{1r}^{A_{1r} + 1} B_{2r}^{A_{2r}} B_{3r}^{A_{3r}}} B(A_{0r}, C_{0r}) B(B_{0r}, A_{0r} + C_{0r}) \right\}^2}{\left\{ Q_r^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^i \binom{n-r}{i} \binom{i}{j} \frac{\Gamma(A_{1r} + 2) \Gamma(A_{2r}) \Gamma(A_{3r})}{B_{1r}^{A_{1r} + 2} B_{2r}^{A_{2r}} B_{3r}^{A_{3r}}} B(A_{0r}, C_{0r}) B(B_{0r}, A_{0r} + C_{0r}) \right\}^2} \quad \dots(43)$$

$$\rho(\hat{p}_{2r}) = 1 - \frac{\left\{ Q_r^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^i \binom{n-r}{i} \binom{i}{j} \frac{\Gamma(A_{1r}) \Gamma(A_{2r} + 1) \Gamma(A_{3r})}{B_{1r}^{A_{1r}} B_{2r}^{A_{2r} + 1} B_{3r}^{A_{3r}}} B(A_{0r}, C_{0r}) B(B_{0r}, A_{0r} + C_{0r}) \right\}^2}{\left\{ Q_r^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^i \binom{n-r}{i} \binom{i}{j} \frac{\Gamma(A_{1r}) \Gamma(A_{2r} + 2) \Gamma(A_{3r})}{B_{1r}^{A_{1r}} B_{2r}^{A_{2r} + 2} B_{3r}^{A_{3r}}} B(A_{0r}, C_{0r}) B(B_{0r}, A_{0r} + C_{0r}) \right\}^2} \quad \dots(44)$$

$$\rho(\hat{p}_{3r}) = 1 - \frac{\left\{ Q_r^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^i \binom{n-r}{i} \binom{i}{j} \frac{\Gamma(A_{1r}) \Gamma(A_{2r}) \Gamma(A_{3r} + 1)}{B_{1r}^{A_{1r}} B_{2r}^{A_{2r}} B_{3r}^{A_{3r} + 1}} B(A_{0r}, C_{0r}) B(B_{0r}, A_{0r} + C_{0r}) \right\}^2}{\left\{ Q_r^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^i \binom{n-r}{i} \binom{i}{j} \frac{\Gamma(A_{1r}) \Gamma(A_{2r}) \Gamma(A_{3r} + 2)}{B_{1r}^{A_{1r}} B_{2r}^{A_{2r}} B_{3r}^{A_{3r} + 2}} B(A_{0r}, C_{0r}) B(B_{0r}, A_{0r} + C_{0r}) \right\}^2} \quad \dots(45)$$

$$\rho(\hat{p}_{4r}) = 1 - \frac{\left\{ Q_r^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^i \binom{n-r}{i} \binom{i}{j} \frac{\Gamma(A_{1r}) \Gamma(A_{2r}) \Gamma(A_{3r})}{B_{1r}^{A_{1r}} B_{2r}^{A_{2r}} B_{3r}^{A_{3r}}} B(B_{0r}, C_{0r}) B(A_{0r} + 1, B_{0r} + C_{0r}) \right\}^2}{\left\{ Q_r^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^i \binom{n-r}{i} \binom{i}{j} \frac{\Gamma(A_{1r}) \Gamma(A_{2r}) \Gamma(A_{3r})}{B_{1r}^{A_{1r}} B_{2r}^{A_{2r}} B_{3r}^{A_{3r}}} B(B_{0r}, C_{0r}) B(A_{0r} + 2, B_{0r} + C_{0r}) \right\}^2} \quad \dots(46)$$

$$\rho(\hat{p}_{5r}) = 1 - \frac{\left\{ Q_r^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^i \binom{n-r}{i} \binom{i}{j} \frac{\Gamma(A_{1r}) \Gamma(A_{2r}) \Gamma(A_{3r})}{B_{1r}^{A_{1r}} B_{2r}^{A_{2r}} B_{3r}^{A_{3r}}} B(A_{0r}, C_{0r}) B(B_{0r} + 1, A_{0r} + C_{0r}) \right\}^2}{\left\{ Q_r^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^i \binom{n-r}{i} \binom{i}{j} \frac{\Gamma(A_{1r}) \Gamma(A_{2r}) \Gamma(A_{3r})}{B_{1r}^{A_{1r}} B_{2r}^{A_{2r}} B_{3r}^{A_{3r}}} B(A_{0r}, C_{0r}) B(B_{0r} + 2, A_{0r} + C_{0r}) \right\}^2} \quad \dots(47)$$

**Elicitation of hyperparameters**

Elicitation is the key task for subjective Bayesian. The whole procedure for quantifying the prior information in the form of prior distribution is precisely known as elicitation. Aslam (2003) proposed different methods of elicitation based on prior predictive distribution for the

elicitation of hyperparameters. In this study, we use the method of elicitation using prior predictive distribution based on predictive probabilities. In this method, confidence levels of the prior predictive are obtained for the particular intervals of the random variables. The set of hyperparameters, for which the difference between the elicited probabilities and the expert predictive probabilities is minimum, is considered.

**Elicitation of hyperparameters using the gamma prior**

For eliciting the hyperparameters, prior predictive distribution (PPD) is used. The PPD for a random variable  $y$  is:

$$p(y) = \int p(y | \psi) \pi_3(\psi) d\psi \quad \dots(48)$$

$$p(y) = \frac{1}{(a + b + c)} \left[ \frac{aa_1 b_1^{a_1} y^{-2}}{(b_1 + y^{-1})^{a_1 + 1}} + \frac{ba_2 b_2^{a_2} y^{-2}}{(b_2 + y^{-1})^{a_2 + 1}} + \frac{ca_3 b_3^{a_3} y^{-2}}{(b_3 + y^{-1})^{a_3 + 1}} \right] \quad \dots(49)$$

We choose the prior predictive probabilities satisfying the laws of probability, to elicit the hyperparameters of the prior density. Using the prior predictive distribution, we consider the nine intervals (0,1), (1,2), (2,3), (3,4), (4,5), (5,6), (6,7), (7,8) and (8,9) with probabilities 0.67, 0.13, 0.06, 0.03, 0.02, 0.01, 0.01, 0.008 and 0.006, respectively given an expert opinion.

For eliciting the hyperparameters  $a_1, a_2, a_3, b_1, b_2, b_3, a, b$  and  $c$ , the equations are simultaneously solved through the computer programme developed in SAS package using the ‘PROC SYSLIN’ command; the values of the hyperparameters are found to be 1.7388, 0.4722, 0.3528, 1.0779, 0.1469, 0.0737, 0.3695, 1.4360 and 8.8749, respectively.

**Limiting expressions**

Letting  $t \rightarrow \infty$ , all the observations that are incorporated in our analysis are uncensored and therefore,  $r$  tends to  $n$ ,  $r_1$  tends to the unknown  $n_1$ ,  $r_2$  tends to the unknown  $n_2$  and  $r_3$  tends to the unknown  $n_3$ . As a result, the amount of information contained in the sample is increasing, which consequently results in the reduction of the variances of the estimates. The limiting (complete sample) expressions for Bayes estimators and posterior risks using the UP, JP and IP under SELF, PLF and DLF are given in Tables 1 – 4. The properties of Bayes estimators and their corresponding posterior risks using complete sampling under SELF are given in Table VII in appendix.

**Table 1:** Limiting expressions for the Bayes estimators as  $t \rightarrow \infty$  using the UP, JP and GP under SELF, PLF and DLF

Loss function	Prior	$\beta_1$	$\beta_2$	$\beta_3$	$p_1$	$p_2$
SELF	UP	$\frac{n_1 + 1}{\sum_{k=1}^{n_1} x_{1k}^{-1}}$	$\frac{n_2 + 1}{\sum_{k=1}^{n_2} x_{2k}^{-1}}$	$\frac{n_3 + 1}{\sum_{k=1}^{n_3} x_{3k}^{-1}}$	$\frac{n_1 + 1}{n + 3}$	$\frac{n_2 + 1}{n + 3}$
	JP	$\frac{n_1}{\sum_{k=1}^{n_1} x_{1k}^{-1}}$	$\frac{n_2}{\sum_{k=1}^{n_2} x_{2k}^{-1}}$	$\frac{n_3}{\sum_{k=1}^{n_3} x_{3k}^{-1}}$	$\frac{n_1 + 1}{n + 3}$	$\frac{n_2 + 1}{n + 3}$
	GP	$\frac{n_1 + a_1}{\sum_{k=1}^{n_1} x_{1k}^{-1} + b_1}$	$\frac{n_2 + a_2}{\sum_{k=1}^{n_2} x_{2k}^{-1} + b_2}$	$\frac{n_3 + a_3}{\sum_{k=1}^{n_3} x_{3k}^{-1} + b_3}$	$\frac{n_1 + a}{n + a + b + c}$	$\frac{n_2 + b}{n + a + b + c}$
PLF	UP	$\frac{[(n_1 + 1)(n_1 + 2)]^{1/2}}{\sum_{k=1}^{n_1} x_{1k}^{-1}}$	$\frac{[(n_2 + 1)(n_2 + 2)]^{1/2}}{\sum_{k=1}^{n_2} x_{2k}^{-1}}$	$\frac{[(n_3 + 1)(n_3 + 2)]^{1/2}}{\sum_{k=1}^{n_3} x_{3k}^{-1}}$	$\left[\frac{(n_1 + 1)(n_1 + 2)}{(n + 3)(n + 4)}\right]^{1/2}$	$\left[\frac{(n_2 + 1)(n_2 + 2)}{(n + 3)(n + 4)}\right]^{1/2}$
	JP	$\frac{[n_1(n_1 + 1)]^{1/2}}{\sum_{k=1}^{n_1} x_{1k}^{-1}}$	$\frac{[n_2(n_2 + 1)]^{1/2}}{\sum_{k=1}^{n_2} x_{2k}^{-1}}$	$\frac{[n_3(n_3 + 1)]^{1/2}}{\sum_{k=1}^{n_3} x_{3k}^{-1}}$	$\left[\frac{(n_1 + 1)(n_1 + 2)}{(n + 3)(n + 4)}\right]^{1/2}$	$\left[\frac{(n_2 + 1)(n_2 + 2)}{(n + 3)(n + 4)}\right]^{1/2}$
	GP	$\left[\frac{(n_1 + a_1)}{(n_1 + a_1 + 1)}\right]^{1/2} \frac{1}{\sum_{k=1}^{n_1} x_{1k}^{-1} + b_1}$	$\left[\frac{(n_2 + a_2)}{(n_2 + a_2 + 1)}\right]^{1/2} \frac{1}{\sum_{k=1}^{n_2} x_{2k}^{-1} + b_2}$	$\left[\frac{(n_3 + a_3)}{(n_3 + a_3 + 1)}\right]^{1/2} \frac{1}{\sum_{k=1}^{n_3} x_{3k}^{-1} + b_3}$	$\left[\frac{(n_1 + a)}{(n + a + b + c)}\right]^{1/2} \frac{1}{(n + a + b + c + 1)}$	$\left[\frac{(n_2 + b)}{(n + a + b + c)}\right]^{1/2} \frac{1}{(n + a + b + c + 1)}$
DLF	UP	$\frac{n_1 + 2}{\sum_{k=1}^{n_1} x_{1k}^{-1}}$	$\frac{n_2 + 2}{\sum_{k=1}^{n_2} x_{2k}^{-1}}$	$\frac{n_3 + 2}{\sum_{k=1}^{n_3} x_{3k}^{-1}}$	$\frac{n_1 + 2}{n + 4}$	$\frac{n_2 + 2}{n + 4}$
	JP	$\frac{n_1 + 1}{\sum_{k=1}^{n_1} x_{1k}^{-1}}$	$\frac{n_2 + 1}{\sum_{k=1}^{n_2} x_{2k}^{-1}}$	$\frac{n_3 + 1}{\sum_{k=1}^{n_3} x_{3k}^{-1}}$	$\frac{n_1 + 2}{n + 4}$	$\frac{n_2 + 2}{n + 4}$
	GP	$\frac{n_1 + a_1 + 1}{\sum_{k=1}^{n_1} x_{1k}^{-1} + b_1}$	$\frac{n_2 + a_2 + 1}{\sum_{k=1}^{n_2} x_{2k}^{-1} + b_2}$	$\frac{n_3 + a_3 + 1}{\sum_{k=1}^{n_3} x_{3k}^{-1} + b_3}$	$\frac{n_1 + a + 1}{n + a + b + c}$	$\frac{n_2 + b + 1}{n + a + b + c}$

**Table 2:** Limiting expressions for the posterior risks as  $t \rightarrow \infty$  using the UP, JP and GP under SELF

Parameters	UP	JP	GP
$\beta_1$	$\frac{n_1 + 1}{\left(\sum_{k=1}^{n_1} x_{1k}^{-1}\right)^2}$	$\frac{n_1}{\left(\sum_{k=1}^{n_1} x_{1k}^{-1}\right)^2}$	$\frac{n_1 + a_1}{\left(\sum_{k=1}^{n_1} x_{1k}^{-1} + b_1\right)^2}$
$\beta_2$	$\frac{n_2 + 1}{\left(\sum_{k=1}^{n_2} x_{2k}^{-1}\right)^2}$	$\frac{n_2}{\left(\sum_{k=1}^{n_2} x_{2k}^{-1}\right)^2}$	$\frac{n_2 + a_2}{\left(\sum_{k=1}^{n_2} x_{2k}^{-1} + b_2\right)^2}$
$\beta_3$	$\frac{n_3 + 1}{\left(\sum_{k=1}^{n_3} x_{3k}^{-1}\right)^2}$	$\frac{n_3}{\left(\sum_{k=1}^{n_3} x_{3k}^{-1}\right)^2}$	$\frac{n_3 + a_3}{\left(\sum_{k=1}^{n_3} x_{3k}^{-1} + b_3\right)^2}$
$p_1$	$\frac{(n_1 + 1)(n_2 + n_3 + 2)}{(n + 3)^2(n + 4)}$	$\frac{(n_1 + 1)(n_2 + n_3 + 2)}{(n + 3)^2(n + 4)}$	$\frac{(n_1 + a)(n_2 + n_3 + b + c)}{(n + a + b + c)^2(n + a + b + c + 1)}$
$p_2$	$\frac{(n_2 + 1)(n_1 + n_3 + 2)}{(n + 3)^2(n + 4)}$	$\frac{(n_2 + 1)(n_1 + n_3 + 2)}{(n + 3)^2(n + 4)}$	$\frac{(n_2 + b)(n_1 + n_3 + a + c)}{(n + a + b + c)^2(n + a + b + c + 1)}$

**Table 3:** Limiting expressions for the posterior risks as  $t \rightarrow \infty$  using the UP, JP and GP under PLF

Parameters	UP	JP	GP
$\beta_1$	$\frac{2(n_1 + 1)}{\left(\sum_{k=1}^{n_1} x_{1k}^{-1}\right)} \left\{ \frac{(n_1 + 2)^{1/2}}{(n_1 + 1)^{1/2}} - 1 \right\}$	$\frac{2n_1}{\left(\sum_{k=1}^{n_1} x_{1k}^{-1}\right)} \left\{ \frac{(n_1 + 1)^{1/2}}{(n_1)^{1/2}} - 1 \right\}$	$\frac{2(n_1 + a_1)}{\left(\sum_{k=1}^{n_1} x_{1k}^{-1} + b_1\right)} \left\{ \frac{(n_1 + a_1 + 1)^{1/2}}{(n_1 + a_1)^{1/2}} - 1 \right\}$
$\beta_2$	$\frac{2(n_2 + 1)}{\left(\sum_{k=1}^{n_2} x_{2k}^{-1}\right)} \left\{ \frac{(n_2 + 2)^{1/2}}{(n_2 + 1)^{1/2}} - 1 \right\}$	$\frac{2n_2}{\left(\sum_{k=1}^{n_2} x_{2k}^{-1}\right)} \left\{ \frac{(n_2 + 1)^{1/2}}{(n_2)^{1/2}} - 1 \right\}$	$\frac{2(n_2 + a_2)}{\left(\sum_{k=1}^{n_2} x_{2k}^{-1} + b_2\right)} \left\{ \frac{(n_2 + a_2 + 1)^{1/2}}{(n_2 + a_2)^{1/2}} - 1 \right\}$
$\beta_3$	$\frac{2(n_3 + 1)}{\left(\sum_{k=1}^{n_3} x_{3k}^{-1}\right)} \left\{ \frac{(n_3 + 2)^{1/2}}{(n_3 + 1)^{1/2}} - 1 \right\}$	$\frac{2n_3}{\left(\sum_{k=1}^{n_3} x_{3k}^{-1}\right)} \left\{ \frac{(n_3 + 1)^{1/2}}{(n_3)^{1/2}} - 1 \right\}$	$\frac{2(n_3 + a_3)}{\left(\sum_{k=1}^{n_3} x_{3k}^{-1} + b_3\right)} \left\{ \frac{(n_3 + a_3 + 1)^{1/2}}{(n_3 + a_3)^{1/2}} - 1 \right\}$
$p_1$	$\frac{2(n_1 + 1)}{(n + 3)} \left\{ \frac{\left(\frac{(n_1 + 2)}{(n_1 + 1)}\right)^{1/2}}{\left(\frac{(n + 4)}{(n + 3)}\right)^{1/2}} - 1 \right\}$	$\frac{2(n_1 + 1)}{(n + 3)} \left\{ \frac{\left(\frac{(n_1 + 2)}{(n_1 + 1)}\right)^{1/2}}{\left(\frac{(n + 4)}{(n + 3)}\right)^{1/2}} - 1 \right\}$	$\frac{2(n_1 + a)}{(n + a + b + c)} \left\{ \frac{\left(\frac{(n_1 + a + 1)}{(n_1 + a)}\right)^{1/2}}{\left(\frac{(n + a + b + c + 1)}{(n + a + b + c)}\right)^{1/2}} - 1 \right\}$
$p_2$	$\frac{2(n_2 + 1)}{(n + 3)} \left\{ \frac{\left(\frac{(n_2 + 2)}{(n_2 + 1)}\right)^{1/2}}{\left(\frac{(n + 4)}{(n + 3)}\right)^{1/2}} - 1 \right\}$	$\frac{2(n_2 + 1)}{(n + 3)} \left\{ \frac{\left(\frac{(n_2 + 2)}{(n_2 + 1)}\right)^{1/2}}{\left(\frac{(n + 4)}{(n + 3)}\right)^{1/2}} - 1 \right\}$	$\frac{2(n_2 + b)}{(n + a + b + c)} \left\{ \frac{\left(\frac{(n_2 + b + 1)}{(n_2 + b)}\right)^{1/2}}{\left(\frac{(n + a + b + c + 1)}{(n + a + b + c)}\right)^{1/2}} - 1 \right\}$

**Table 4:** Limiting expressions for the posterior risks as  $t \rightarrow \infty$  using the UP, JP and GP under DLF

Parameters	UP	JP	Gamma prior
$\beta_1$	$\frac{1}{n_1 + 2}$	$\frac{1}{n_1 + 1}$	$\frac{1}{n_1 + a_1 + 1}$
$\beta_2$	$\frac{1}{n_2 + 2}$	$\frac{1}{n_2 + 1}$	$\frac{1}{n_2 + a_2 + 1}$
$\beta_3$	$\frac{1}{n_3 + 2}$	$\frac{1}{n_3 + 1}$	$\frac{1}{n_3 + a_3 + 1}$
$p_1$	$\frac{(n_2 + n_3 + 2)}{(n_1 + 2)(n + 3)}$	$\frac{(n_2 + n_3 + 2)}{(n_1 + 2)(n + 3)}$	$\frac{(n_2 + n_3 + b + c)}{(n_1 + a + 1)(n + a + b + c)}$
$p_2$	$\frac{(n_1 + n_3 + 2)}{(n_2 + 2)(n + 3)}$	$\frac{(n_1 + n_3 + 2)}{(n_2 + 2)(n + 3)}$	$\frac{(n_1 + n_3 + a + c)}{(n_2 + b + 1)(n + a + b + c)}$

**Table 5:** Bayes estimates (BEs) and posterior risks (PRs) of 3-component mixture of Gumbel type-II distributions using the UP, JP, and GP under SELF, PLF and DLF with Crowder *et al.* (1994) mixture data

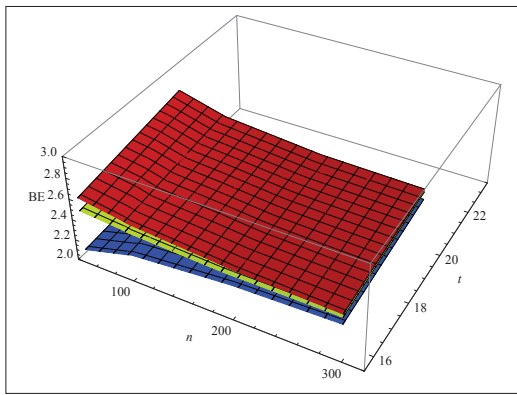
Prior	Loss functions		$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{p}_1$	$\hat{p}_2$
UP	SELF	BE	3.00174	2.43043	1.99007	0.19558	0.34294
		PR	0.676233	0.247192	0.121833	0.002011	0.002962
	PLF	BE	3.11234	2.48076	2.02045	0.20066	0.34723
		PR	0.221205	0.100665	0.060756	0.010148	0.008582
	DLF	BE	3.22702	2.53213	2.05129	0.20586	0.35158
		PR	0.069811	0.040167	0.029845	0.049934	0.024564
JP	SELF	BE	2.79327	2.34029	1.93776	0.19715	0.34295
		PR	0.620029	0.236873	0.118651	0.002034	0.002956
	PLF	BE	2.90213	2.39036	1.96814	0.20225	0.34723
		PR	0.217730	0.100144	0.060755	0.010184	0.008567
	DLF	BE	3.01524	2.44151	1.99899	0.20747	0.35157
		PR	0.073617	0.041456	0.030631	0.049718	0.024520
GP	SELF	BE	2.63492	2.40087	1.91168	0.17412	0.31509
		PR	0.469892	0.242927	0.113700	0.001672	0.002551
	PLF	BE	2.72262	2.45094	1.94119	0.17886	0.31911
		PR	0.175413	0.100138	0.059021	0.009473	0.008045
	DLF	BE	2.81325	2.50206	1.97115	0.18372	0.32319
		PR	0.063390	0.040440	0.030173	0.052265	0.025053

**RESULTS AND DISCUSSION**

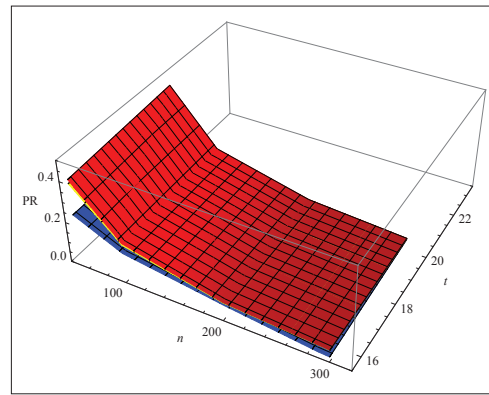
In order to obtain and investigate the properties of the derived estimators, simulation analysis has been carried out. Simulation study is a flexible methodology to illustrate the properties of the Bayes estimators of the 3-component mixture of the Gumbel type-II distributions using the UP, JP and GP under SELF, PLF and DLF in terms of different sample sizes and test termination times. Random samples of size  $n = 50, 100, 200$  and  $300$  have been generated from the Gumbel type-II mixture model by using different values of the vector of the parameters =  $\{(2, 3, 4, 0.20, 0.60), (3, 3, 3, 0.40, 0.40)\}$ .

For a fixed sample size, test termination time and set of parameters, the observations are randomly taken from the first, second and third component density. The observations, which are greater than the fixed ‘t’ were

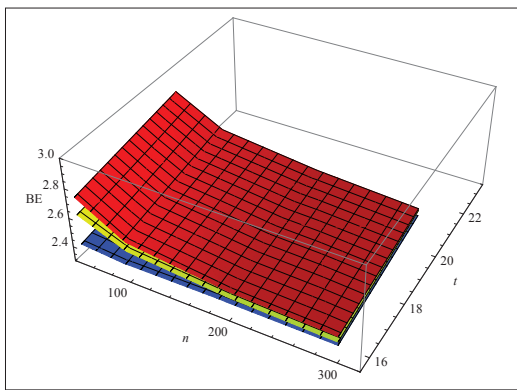
declared as censored observations. For each t, only failures were identified either as a member of subpopulation I, subpopulation II or subpopulation III. This process was replicated 1000 times using a Mathematica 10.0 programme. For each of the 1000 samples, the Bayes estimators and posterior risks were computed and the results are presented in the appendix from Tables I to VII. The simulation study provides some interesting properties of the Bayes estimates. The properties are highlighted in terms of sample sizes, the size of mixing proportions parameters, different loss functions and censoring rates. Estimates are obtained by using the UP, JP and GP. The same results have been examined in Figures 1 – 10 that are based on simulation analysis tables corresponding to the different prior distributions and various loss functions. In Figures 1 – 4, the UP, JP and GP are represented by red, yellow and blue colours, while in Figures 5 – 10, SELF, PLF and DLF are represented by red, yellow and blue colours, respectively.



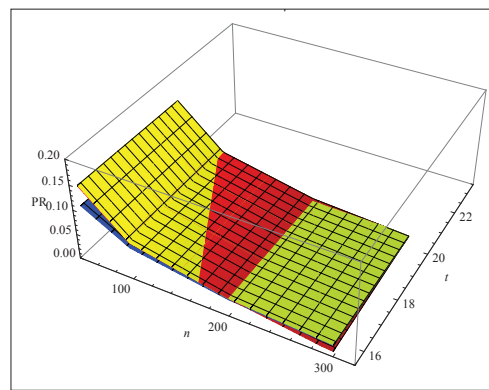
**Figure 1:** BEs for  $\beta_1$  under SELF



**Figure 2:** PRs for  $\beta_1$  under SELF



**Figure 3:** BEs for  $\beta_2$  under PLF



**Figure 4:** PRs for  $\beta_2$  under PLF

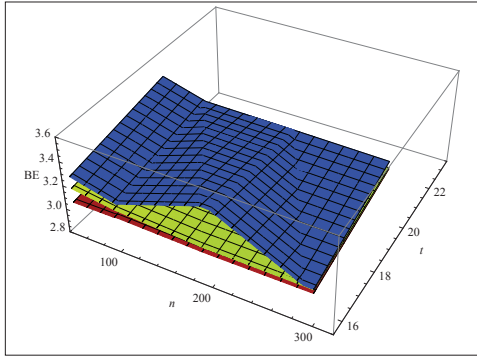


Figure 5: BEs for  $\beta_3$  using GP

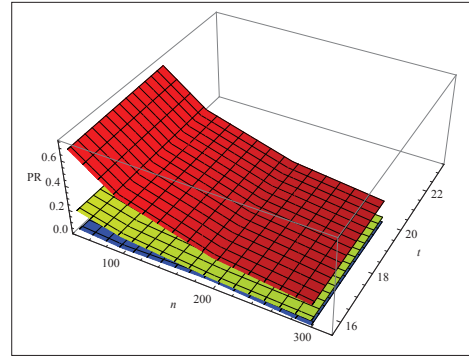


Figure 6: PRs for  $\beta_3$  using GP

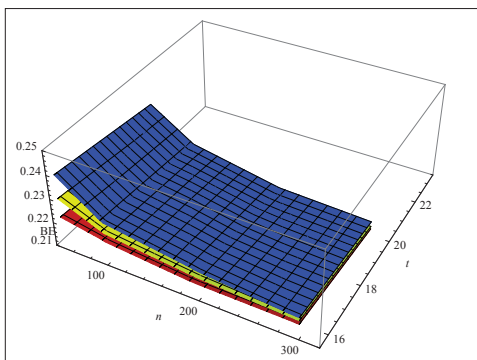


Figure 7: BEs for  $P_1$  using UP

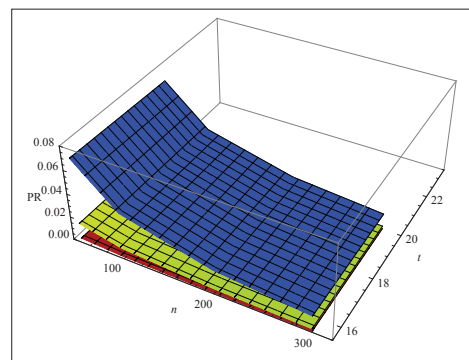


Figure 8: PRs for  $P_1$  using UP

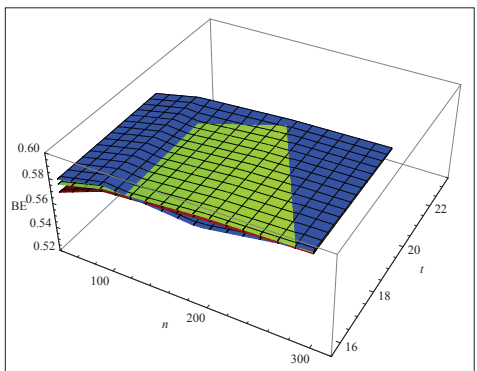


Figure 9: BEs for  $P_2$  using GP

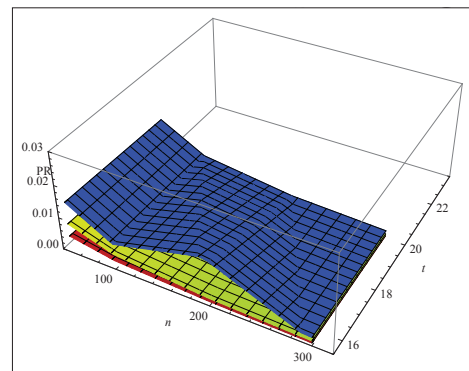


Figure 10: PRs for  $P_2$  using GP

It is observed that the PRs of Bayes estimators using different priors and loss functions reduce with an increase in sample size at different test termination times. For a smaller test termination time, the PRs of Bayes estimators are larger than the PRs for a larger test termination time, irrespective of the priors, loss function and sample size. Also, the PRs of Bayes estimators are smaller (larger) for

smaller (larger) component parametric values for each sample size and test termination time considered in the simulation study.

In Table VII in appendix, some results for the complete sample under SELF are given, which generally highlights the importance of incorporating full information into the

sample by producing Bayes estimates that do not involve underestimation for any loss function or prior used, and provides smaller posterior risks. The same results have been obtained under PLF and DLF.

### A real life data application

Crowder *et al.* (1994) have reported the data on fibre failure strength, and the breaking strength of fibre sections of lengths 5, 12, 30 and 45 mm. To elucidate the proposed methodology, we have taken the data on 3-component, namely, 5, 12 and 30 mm, respectively. The values are right censored at 4.0 i.e.  $t = 4.0$ . The sample statistics required to evaluate the proposed estimates are as follows:

$$n = 102, r_1 = 15, r_2 = 26, r_3 = 34, r = 75, n - r = 27,$$

$$\sum_{k=1}^{r_1} x_{1k}^{-1} = 4.2612, \quad \sum_{k=1}^{r_2} x_{2k}^{-1} = 8.9087, \quad \sum_{k=1}^{r_3} x_{3k}^{-1} = 14.2608.$$

Bayes estimates and Bayes posterior risks (BPRs) using the UP, JP and GP under SELF, PLF and DLF are given in Table 5.

It is noted that the results gained from real data are compatible with simulation results. The results declare that the execution of the informative prior is better than the non-informative priors. It is also examined that execution of DLF is preferred for estimating the component parameters, while SELF is better for estimating the proportion parameters.

### CONCLUSION

As far as the problem of selecting the most suitable prior is concerned, it can be seen that the informative prior (GP) is the best prior among the assumed priors. On the other hand, DLF is observed performing better than PLF and SELF for estimating component parameters, whereas for estimating proportion parameters, SELF is observed superior to PLF and DLF. Therefore, the informative prior has a clear edge over the non-informative prior. It is to be noted that the selection of the best prior (loss function) for a given loss function (prior) is made on PRs associated with it. Also, the selection of the best prior and loss function does not depend on the sample size and test termination time. The findings of this study are compatible with the findings described by Feroz and Aslam (2012) in Bayesian estimation of 2-component mixture of Gumbel Type-II distributions and are useful for analysts looking to model lifetime data, where the causes of failure are more than one.

From simulated results, we observed that an increase in the sample size and test termination time provides better Bayes estimators. Furthermore, as the sample size increases (decreases) the posterior risks of Bayes estimators decreases (increases) for a fixed test termination time. Also, the DLF is observed as a suitable choice for estimating component parameters, and SELF is preferable for estimating proportion parameters. Finally, we conclude that the GP is the suitable prior in order to estimate the component parameters. When SELF is used, the GP is an appropriate prior for proportion parameters. The same pattern is observed for JP when non-informative priors are considered.

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APPENDIX

**Table I:** Bayes estimates (BEs) and posterior risks (PRs) of 3-component mixture of Gumbel type-II distributions using the UP under SELF, PLF and DLF with  $\beta_1 = 2, \beta_2 = 3, \beta_3 = 4, p_1 = 0.20, p_2 = 0.60, t = 15, 20$

t	n	Loss functions		UP				
				$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{p}_1$	$\hat{p}_2$
15	50	SELF	BE	2.05963	2.55289	3.67339	0.22425	0.58263
			PR	0.482935	0.264136	1.79543	0.003897	0.005405
		PLF	BE	2.19242	2.63541	3.91421	0.23213	0.58712
			PR	0.214497	0.101558	0.437589	0.017182	0.009336
		DLF	BE	2.28623	2.67729	4.14929	0.24211	0.59072
			PR	0.094131	0.038059	0.107369	0.072757	0.015954
	100	SELF	BE	1.9083	2.47895	3.37552	0.22084	0.58924
			PR	0.207845	0.12508	0.748593	0.002019	0.002821
		PLF	BE	1.94512	2.52087	3.4678	0.22440	0.59252
			PR	0.102426	0.049940	0.208573	0.009078	0.004782
		DLF	BE	2.0144	2.56464	3.55651	0.22876	0.59437
			PR	0.051802	0.019695	0.058730	0.040271	0.008087
	200	SELF	BE	1.8058	2.45322	3.17286	0.21748	0.59484
			PR	0.093528	0.061420	0.331186	0.001024	0.001441
		PLF	BE	1.84137	2.4676	3.22743	0.22039	0.59524
			PR	0.050347	0.024795	0.101088	0.004668	0.002420
		DLF	BE	1.85144	2.48367	3.28593	0.22325	0.59672
			PR	0.027120	0.010030	0.031212	0.0211	0.004069
	300	SELF	BE	1.77416	2.43709	3.11848	0.21716	0.59595
			PR	0.059966	0.040370	0.212681	0.000686	0.000965
		PLF	BE	1.78096	2.44789	3.17044	0.21873	0.59607
			PR	0.033017	0.016504	0.067174	0.003149	0.001623
		DLF	BE	1.80725	2.46302	3.18745	0.21992	0.59787
			PR	0.018491	0.006724	0.021136	0.014418	0.002714
20	50	SELF	BE	2.19255	2.75901	3.98405	0.22086	0.58229
			PR	0.529995	0.294373	1.96752	0.003677	0.005170
		PLF	BE	2.29534	2.81129	4.16831	0.22801	0.58747
			PR	0.215926	0.102638	0.434676	0.016384	0.008858
		DLF	BE	2.37214	2.84112	4.41512	0.23687	0.59125
			PR	0.090964	0.036132	0.100275	0.070849	0.015106
	100	SELF	BE	1.97456	2.62072	3.63898	0.21549	0.59029
			PR	0.216296	0.132854	0.806272	0.001883	0.002682
		PLF	BE	2.00635	2.65615	3.68027	0.21964	0.59295
			PR	0.101846	0.049947	0.20561	0.008666	0.004541
		DLF	BE	2.08985	2.70458	3.80788	0.22497	0.59449
			PR	0.050019	0.018781	0.055241	0.039135	0.007705
	200	SELF	BE	1.88328	2.60844	3.39904	0.21392	0.59515
			PR	0.097850	0.065837	0.352952	0.000958	0.001367
		PLF	BE	1.91467	2.60709	3.4868	0.21558	0.59594
			PR	0.050685	0.024890	0.101762	0.004461	0.002299
		DLF	BE	1.9436	2.62831	3.51094	0.21776	0.59786
			PR	0.026292	0.009506	0.02904	0.020641	0.003845
	300	SELF	BE	1.85822	2.57683	3.34688	0.21291	0.59656
			PR	0.063635	0.042874	0.228372	0.000642	0.000918
		PLF	BE	1.87119	2.58445	3.39781	0.21401	0.59728
			PR	0.033540	0.016527	0.067136	0.003005	0.001538
		DLF	BE	1.88059	2.60682	3.42132	0.21608	0.59795
			PR	0.017818	0.006393	0.019735	0.013989	0.002580

**Table II:** Bayes estimates (BEs) and posterior risks (PRs) of 3-component mixture of Gumbel type-II distributions using the JP under SELF, PLF and DLF with  $\beta_1 = 2, \beta_2 = 3, \beta_3 = 4, p_1 = 0.20, p_2 = 0.60, t = 15, 20$

t	n	Loss functions		JP				
				$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{p}_1$	$\hat{p}_2$
15	50	SELF	BE	1.91359	2.45155	3.2509	0.22435	0.58195
			PR	0.473989	0.252987	1.60348	0.003897	0.005413
		PLF	BE	1.97287	2.51875	3.52761	0.23276	0.58566
			PR	0.212444	0.10088	0.44154	0.01714	0.00935
		DLF	BE	2.12995	2.56425	3.67571	0.24057	0.59092
			PR	0.104298	0.0393711	0.119158	0.0731143	0.0158882
	100	SELF	BE	1.79531	2.43366	3.14736	0.22006	0.58906
			PR	0.194814	0.122887	0.687328	0.002009	0.002815
		PLF	BE	1.84124	2.48089	3.21993	0.224475	0.59115
			PR	0.102013	0.050157	0.204803	0.009063	0.004791
		DLF	BE	1.86886	2.50321	3.38474	0.22953	0.59462
			PR	0.054494	0.020077	0.062901	0.040128	0.008080
	200	SELF	BE	1.75336	2.4217	3.08974	0.21828	0.59340
			PR	0.090090	0.060389	0.321638	0.001023	0.001437
		PLF	BE	1.7756	2.43406	3.13438	0.22007	0.59628
			PR	0.050067	0.024679	0.10183	0.004675	0.002416
		DLF	BE	1.79672	2.4459	3.18386	0.22295	0.59565
			PR	0.027906	0.010143	0.031971	0.021133	0.004085
	300	SELF	BE	1.73697	2.43213	3.06366	0.21697	0.59573
			PR	0.058619	0.040471	0.209437	0.000685	0.000965
		PLF	BE	1.75734	2.42595	3.0954	0.21841	0.59728
			PR	0.033207	0.016422	0.067263	0.003148	0.001617
		DLF	BE	1.77108	2.44365	3.12952	0.2213	0.59661
			PR	0.018707	0.006779	0.021598	0.014293	0.002727
20	50	SELF	BE	1.98608	2.62234	3.58365	0.22047	0.58209
			PR	0.488933	0.276524	1.7856	0.003668	0.005166
		PLF	BE	2.05258	2.67762	3.6995	0.22897	0.58573
			PR	0.210662	0.101463	0.42616	0.016341	0.008885
		DLF	BE	2.16879	2.74603	3.93224	0.23758	0.59052
			PR	0.100202	0.037637	0.112275	0.070830	0.015206
	100	SELF	BE	1.85242	2.58929	3.40073	0.21547	0.59044
			PR	0.200264	0.132206	0.748278	0.001886	0.002686
		PLF	BE	1.92981	2.60099	3.52903	0.22024	0.59309
			PR	0.102819	0.049756	0.209779	0.008652	0.004536
		DLF	BE	1.94744	2.65098	3.63798	0.22414	0.59434
			PR	0.052889	0.019140	0.058144	0.039334	0.007710
	200	SELF	BE	1.82224	2.56221	3.31138	0.21249	0.59504
			PR	0.094558	0.064152	0.342771	0.000954	0.001367
		PLF	BE	1.84911	2.58254	3.38609	0.21616	0.59609
			PR	0.050298	0.024948	0.102469	0.004472	0.002305
		DLF	BE	1.88254	2.5993	3.41256	0.21774	0.59728
			PR	0.027018	0.009611	0.029842	0.020653	0.003856
	300	SELF	BE	1.82188	2.56824	3.30343	0.21249	0.59504
			PR	0.094558	0.064152	0.226589	0.000640	0.000917
		PLF	BE	1.83324	2.58679	3.30813	0.21407	0.59740
			PR	0.033455	0.016643	0.066773	0.003005	0.001538
		DLF	BE	1.85456	2.58981	3.37147	0.21540	0.59811
			PR	0.018178	0.006425	0.020056	0.014029	0.002576

**Table III:** Bayes estimates (BEs) and posterior risks (PRs) of 3-component mixture of Gumbel type-II distributions using the UP under SELF, PLF and DLF with  $\beta_1 = 3, \beta_2 = 3, \beta_3 = 3, p_1 = 0.40, p_2 = 0.40, t = 15, 20$

t	n	Loss functions		UP				
				$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{p}_1$	$\hat{p}_2$
15	50	SELF	BE	2.62793	2.64591	2.95677	0.39521	0.39604
			PR	0.421742	0.425329	1.07593	0.005321	0.005323
		PLF	BE	2.70489	2.75525	3.08741	0.40154	0.40333
			PR	0.152277	0.154277	0.321992	0.013436	0.013394
		DLF	BE	2.81627	2.81706	3.24022	0.40897	0.40965
			PR	0.055552	0.055434	0.101097	0.033555	0.033465
	100	SELF	BE	2.5562	2.54079	2.63976	0.39839	0.39763
			PR	0.199723	0.197194	0.427289	0.002797	0.002795
		PLF	BE	2.57602	2.56665	2.74725	0.40174	0.40104
			PR	0.07546	0.075397	0.154999	0.007018	0.007026
		DLF	BE	2.63711	2.63325	2.82425	0.40435	0.40506
			PR	0.029048	0.028992	0.055161	0.017479	0.017425
	200	SELF	BE	2.48608	2.47612	2.54494	0.39995	0.39856
			PR	0.094380	0.093854	0.199052	0.001434	0.001432
		PLF	BE	2.47993	2.49275	2.5724	0.40106	0.40089
			PR	0.037096	0.037296	0.075674	0.003581	0.003582
		DLF	BE	2.51629	2.5166	2.58785	0.40217	0.40237
			PR	0.014923	0.014913	0.029039	0.008956	0.008947
	300	SELF	BE	2.45501	2.44369	2.49606	0.39848	0.39954
			PR	0.061591	0.060864	0.126877	0.000961	0.000962
		PLF	BE	2.47862	2.47159	2.53897	0.40093	0.40174
			PR	0.010058	0.010038	0.019690	0.006035	0.006015
		DLF	BE	2.47185	2.46913	2.5463	0.40141	0.40250
			PR	0.010050	0.010024	0.019822	0.006025	0.005998
20	50	SELF	BE	2.82852	2.8412	3.12293	0.39401	0.39794
			PR	0.465625	0.466401	1.14887	0.005075	0.005093
		PLF	BE	2.90408	2.88555	3.26087	0.40227	0.40279
			PR	0.154803	0.153578	0.32237	0.012791	0.012777
		DLF	BE	2.99341	2.93687	3.43063	0.40983	0.40748
			PR	0.052295	0.052619	0.095686	0.031563	0.031883
	100	SELF	BE	2.70475	2.70685	2.7989	0.39821	0.39638
			PR	0.21215	0.213714	0.453306	0.002661	0.002656
		PLF	BE	2.72495	2.71639	2.90887	0.40162	0.40074
			PR	0.075495	0.075436	0.154702	0.006650	0.006660
		DLF	BE	2.76686	2.76649	2.99525	0.40457	0.40399
			PR	0.027602	0.027633	0.052424	0.016619	0.016649
	200	SELF	BE	2.61974	2.61182	2.6998	0.39916	0.39873
			PR	0.099579	0.099068	0.211307	0.001361	0.001361
		PLF	BE	2.63904	2.63855	2.73	0.40132	0.39985
			PR	0.037443	0.037568	0.075892	0.003404	0.003412
		DLF	BE	2.66335	2.66555	2.76496	0.401336	0.40263
			PR	0.014202	0.014156	0.027536	0.008544	0.008498
	300	SELF	BE	2.59774	2.59539	2.63784	0.39879	0.39944
			PR	0.065525	0.065301	0.134688	0.000916	0.000916
		PLF	BE	2.61575	2.59533	2.6644	0.40040	0.40008
			PR	0.024985	0.024815	0.050119	0.002290	0.002292
		DLF	BE	2.62345	2.62279	2.69464	0.40226	0.40097
			PR	0.009508	0.009539	0.018746	0.005699	0.005730

**Table IV:** Bayes estimates (BEs) and posterior risks (PRs) of 3-component mixture of Gumbel type-II distributions using the JP under SELF, PLF and DLF with  $\beta_1 = 3, \beta_2 = 3, \beta_3 = 3, p_1 = 0.40, p_2 = 0.40, t = 15, 20$

t	n	Loss functions		JP				
				$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{p}_1$	$\hat{p}_2$
15	50	SELF	BE	2.50579	2.51536	2.64102	0.39644	0.39490
			PR	0.402872	0.406845	0.957829	0.005300	0.005292
		PLF	BE	2.57686	2.63748	2.77313	0.40212	0.40141
			PR	0.152588	0.156486	0.318867	0.013379	0.013393
		DLF	BE	2.66136	2.67441	2.93516	0.40854	0.40841
			PR	0.058533	0.058548	0.11116	0.033449	0.033467
	100	SELF	BE	2.46833	2.45947	2.48678	0.39648	0.39748
			PR	0.191831	0.190109	0.396216	0.002786	0.002789
		PLF	BE	2.48161	2.50522	2.60458	0.40145	0.40131
			PR	0.074904	0.075625	0.155267	0.007013	0.007015
		DLF	BE	2.53792	2.53114	2.67258	0.40506	0.40403
			PR	0.029799	0.029891	0.058240	0.017395	0.017480
	200	SELF	BE	2.42514	2.44289	2.4845	0.39850	0.39877
			PR	0.091510	0.092750	0.194863	0.001431	0.001431
		PLF	BE	2.45645	2.45217	2.48111	0.39999	0.40112
			PR	0.0374	0.037233	0.074844	0.003587	0.003580
		DLF	BE	2.47872	2.46598	2.52285	0.40259	0.40242
			PR	0.015137	0.015142	0.029992	0.008945	0.008949
	300	SELF	BE	2.42342	2.41642	2.45373	0.39863	0.39971
			PR	0.060662	0.060112	0.125576	0.000962	0.000962
		PLF	BE	2.44252	2.43282	2.47083	0.40024	0.40036
			PR	0.024837	0.024733	0.049966	0.002409	0.002408
		DLF	BE	2.46121	2.45183	2.49706	0.40183	0.40168
			PR	0.010131	0.010135	0.020166	0.006009	0.006013
20	50	SELF	BE	2.68043	2.6468	2.80766	0.39553	0.39569
			PR	0.44025	0.430049	1.03467	0.005073	0.005075
		PLF	BE	2.74963	2.7575	2.9669	0.40222	0.40176
			PR	0.154661	0.155583	0.324787	0.012783	0.012792
		DLF	BE	2.81269	2.80334	3.14895	0.40840	0.40935
			PR	0.055680	0.055555	0.10659	0.031909	0.031783
	100	SELF	BE	2.5937	2.59228	2.65418	0.39748	0.39741
			PR	0.200978	0.201306	0.429992	0.002662	0.002662
		PLF	BE	2.66345	2.637	2.74595	0.40147	0.40063
			PR	0.076000	0.075421	0.154451	0.003403	0.003408
		DLF	BE	2.71343	2.68486	2.81504	0.40482	0.40401
			PR	0.028407	0.028475	0.055509	0.016624	0.016686
	200	SELF	BE	2.57921	2.59528	2.611	0.39828	0.39922
			PR	0.098256	0.099295	0.203591	0.001362	0.001363
		PLF	BE	2.62219	2.58364	2.65262	0.40010	0.40025
			PR	0.037722	0.037246	0.075879	0.003403	0.003408
		DLF	BE	2.61636	2.61068	2.70208	0.40239	0.40180
			PR	0.014380	0.014401	0.028380	0.008514	0.008534
	300	SELF	BE	2.57043	2.57452	2.59734	0.39887	0.39941
			PR	0.06469	0.064765	0.132917	0.000914	0.000915
		PLF	BE	2.57775	2.58242	2.61386	0.40098	0.40051
			PR	0.024827	0.024906	0.050360	0.002289	0.002290
		DLF	BE	2.60664	2.6071	2.65028	0.40158	0.40194
			PR	0.009625	0.009616	0.019149	0.005721	0.005713

**Table V:** Bayes estimates (BEs) and posterior risks (PRs) of 3-component mixture of Gumbel type-II distributions using the GP under SELF, PLF and DLF with  $\beta_1 = 2, \beta_2 = 3, \beta_3 = 4, a_1 = 1.7388, a_2 = 0.4722, a_3 = 0.3528, b_1 = 1.0779, b_2 = 0.1469, b_3 = 0.0737, a = 0.3695, b = 1.4360, c = 8.8749, p_1 = 0.20, p_2 = 0.60, t = 15, 20$ .

t	n	Loss functions		GP				
				$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{p}_1$	$\hat{p}_2$
15	50	SELF	BE	1.56683	2.40591	3.07652	0.23817	0.57133
			PR	0.241388	0.207639	0.711495	0.003932	0.005267
		PLF	BE	1.66061	2.44352	3.19152	0.24824	0.57807
			PR	0.145569	0.083929	0.226379	0.016248	0.009192
		DLF	BE	1.72694	2.4871	3.28841	0.25575	0.58198
			PR	0.085907	0.034032	0.069234	0.065194	0.015959
	100	SELF	BE	1.65448	2.40522	3.07746	0.22797	0.585222
			PR	0.146583	0.111094	0.46131	0.002038	0.002790
		PLF	BE	1.68095	2.43261	3.14103	0.23219	0.58733
			PR	0.083698	0.045390	0.144118	0.008823	0.004750
		DLF	BE	1.73631	2.44626	3.19767	0.23654	0.59015
			PR	0.049253	0.018571	0.045425	0.037906	0.008087
	200	SELF	BE	1.67789	2.40478	3.0596	0.22234	0.59080
			PR	0.077589	0.057147	0.260907	0.001027	0.001426
		PLF	BE	1.70871	2.40322	3.10043	0.22439	0.59315
			PR	0.045493	0.023430	0.084019	0.004606	0.002412
		DLF	BE	1.78005	2.60856	3.46066	0.24920	0.58346
			PR	0.083875	0.032581	0.066443	0.064252	0.015133
	300	SELF	BE	1.688	2.40726	3.06311	0.22019	0.59429
			PR	0.053211	0.038604	0.184811	0.000689	0.000961
		PLF	BE	1.70816	2.42443	3.09497	0.22206	0.59540
			PR	0.031020	0.016004	0.059690	0.003119	0.001617
		DLF	BE	1.73084	2.42641	3.09362	0.22294	0.59590
			PR	0.018117	0.006588	0.019093	0.014073	0.002718
20	50	SELF	BE	1.63206	2.52061	3.2088	0.23250	0.57498
			PR	0.2555	0.217936	0.749546	0.003695	0.005036
		PLF	BE	1.69767	2.54503	3.34103	0.24197	0.57906
			PR	0.14462	0.083487	0.226435	0.015594	0.008744
		DLF	BE	1.78005	2.60856	3.46066	0.24920	0.58346
			PR	0.083875	0.032581	0.066443	0.064252	0.015133
	100	SELF	BE	1.71018	2.5279	3.26042	0.22321	0.58629
			PR	0.150658	0.116468	0.489156	0.001901	0.002647
		PLF	BE	1.74817	2.55661	3.29384	0.22741	0.58849
			PR	0.084285	0.045499	0.143777	0.008449	0.004518
		DLF	BE	1.7895	2.57899	3.35566	0.23192	0.59098
			PR	0.047566	0.017695	0.043204	0.036876	0.007668
	200	SELF	BE	1.75244	2.52875	3.25501	0.21805	0.59295
			PR	0.081961	0.060124	0.280411	0.000964	0.001359
		PLF	BE	1.77711	2.55312	3.28737	0.21942	0.59456
			PR	0.045908	0.02368	0.084041	0.004408	0.002290
		DLF	BE	1.80335	2.57442	3.34172	0.22243	0.59505
			PR	0.025536	0.009256	0.025407	0.019926	0.003861
	300	SELF	BE	1.7727	2.54839	3.26254	0.21506	0.59560
			PR	0.056899	0.041106	0.196652	0.000643	0.000914
		PLF	BE	1.77372	2.56268	3.2735	0.21699	0.59544
			PR	0.031184	0.016086	0.058896	0.002978	0.001537
		DLF	BE	1.7855	2.57433	3.32566	0.21837	0.59626
			PR	0.017496	0.006262	0.017886	0.013696	0.002579

**Table VI:** Bayes estimates (BEs) and posterior risks (PRs) of 3-component mixture of Gumbel type-II distributions using the GP under SELF, PLF and DLF with  $\beta_1 = 3, \beta_2 = 3, \beta_3 = 3, a_1 = 1.7388, a_2 = 0.4722, a_3 = 0.3528, b_1 = 1.0779, b_2 = 0.1469, b_3 = 0.0737, a = 0.3695, b = 1.4360, c = 8.8749, p_1 = 0.40, p_2 = 0.40, t = 15, 20$

T	n	Loss functions		GP				
				$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{p}_1$	$\hat{p}_2$
15	50	SELF	BE	2.12678	2.38951	2.72082	0.40564	0.39293
			PR	0.254593	0.289911	0.544987	0.005204	0.005144
		PLF	BE	2.18569	2.43408	2.82963	0.41106	0.40064
			PR	0.115597	0.115898	0.191636	0.012760	0.012965
		DLF	BE	2.2665	2.48761	2.89171	0.41782	0.40747
			PR	0.052405	0.047261	0.066818	0.031160	0.032519
	100	SELF	BE	2.25086	2.3792	2.6293	0.40268	0.39659
			PR	0.149246	0.157632	0.319119	0.002756	0.002740
		PLF	BE	2.30602	2.40795	2.66264	0.40645	0.39950
			PR	0.065449	0.064773	0.1162	0.006823	0.006899
		DLF	BE	2.32588	2.46535	2.71347	0.40958	0.40366
			PR	0.028255	0.02671	0.043260	0.016847	0.017258
	200	SELF	BE	2.33068	2.40639	2.53466	0.40174	0.39823
			PR	0.081609	0.084513	0.17035	0.001422	0.001417
		PLF	BE	2.35514	2.41406	2.5509	0.40375	0.39892
			PR	0.03468	0.034658	0.064973	0.003533	0.003561
		DLF	BE	2.36777	2.43104	2.59392	0.40521	0.4023
			PR	0.014684	0.014251	0.025453	0.008765	0.008868
	300	SELF	BE	2.35866	2.40545	2.48941	0.40083	0.39948
			PR	0.056178	0.057160	0.114795	0.000958	0.000956
		PLF	BE	2.37855	2.41126	2.50286	0.40193	0.40030
			PR	0.023701	0.023529	0.045154	0.002388	0.002394
		DLF	BE	2.37596	2.4273	2.54011	0.40338	0.40113
			PR	0.009925	0.009731	0.017937	0.005930	0.005984
20	50	SELF	BE	2.23761	2.48796	2.78189	0.40413	0.39282
			PR	0.26815	0.301329	0.548209	0.004953	0.004903
		PLF	BE	2.29827	2.55371	2.89405	0.41080	0.39812
			PR	0.115619	0.117149	0.188654	0.012185	0.012436
		DLF	BE	2.3717	2.62563	3.00873	0.41742	0.40687
			PR	0.049823	0.045242	0.064752	0.029644	0.030952
	100	SELF	BE	2.39365	2.52706	2.71051	0.40296	0.39629
			PR	0.160617	0.16999	0.327067	0.002632	0.002616
		PLF	BE	2.42274	2.56881	2.78415	0.40611	0.39986
			PR	0.065507	0.065964	0.117108	0.006516	0.006580
		DLF	BE	2.47136	2.59433	2.85391	0.40957	0.40241
			PR	0.026780	0.025501	0.041446	0.015980	0.016465
	200	SELF	BE	2.46999	2.53253	2.64858	0.40142	0.39787
			PR	0.087172	0.089220	0.17707	0.001353	0.001349
		PLF	BE	2.48995	2.55163	2.68829	0.40324	0.40031
			PR	0.034869	0.034753	0.065770	0.003366	0.003381
		DLF	BE	2.50759	2.58732	2.71896	0.40571	0.40082
			PR	0.013930	0.013613	0.024274	0.008318	0.008487
	300	SELF	BE	2.49846	2.53969	2.62927	0.40121	0.39876
			PR	0.059744	0.060617	0.12178	0.000910	0.000908
		PLF	BE	2.51474	2.5473	2.64388	0.40220	0.39991
			PR	0.023784	0.023656	0.045509	0.002271	0.002279
		DLF	BE	2.51968	2.56658	2.67343	0.40341	0.40121
			PR	0.009441	0.009269	0.017166	0.005648	0.005698

**Table VII:** Bayes estimates and posterior risks (in parentheses) under SELF using complete sampling

$\Phi = (\hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3, \hat{p}_1, \hat{p}_2)$					
$\Phi = (2, 3, 4, 0.20, 0.60)$ UP					
n	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{p}_1$	$\hat{p}_2$
50	2.48265 (0.629920)	3.21679 (0.345467)	4.88026 (2.437100)	0.20755 (0.003046)	0.58491 (0.004496)
100	2.22325 (0.248929)	3.09560 (0.159958)	4.44845 (0.996820)	0.20388 (0.001561)	0.59223 (0.002322)
200	2.10990 (0.111342)	3.05416 (0.077774)	4.22293 (0.446620)	0.20197 (0.000790)	0.59606 (0.001180)
300	2.06432 (0.070969)	3.04423 (0.051500)	4.13686 (0.285329)	0.20132 (0.005289)	0.59736 (0.000791)
$\Phi = (\hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3, \hat{p}_1, \hat{p}_2)$					
$\Phi = (2, 3, 4, 0.20, 0.60)$ JP					
n	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{p}_1$	$\hat{p}_2$
50	2.21951 (0.550923)	3.08711 (0.328661)	4.36265 (0.550923)	0.20755 (0.003046)	0.58491 (0.004496)
100	2.10056 (0.235037)	3.06126 (0.158854)	4.17918 (0.235037)	0.20388 (0.001561)	0.59223 (0.002322)
200	2.04042 (0.106829)	3.02665 (0.076988)	4.12896 (0.106829)	0.20197 (0.000790)	0.59606 (0.001180)
300	2.20698 (0.069618)	3.01377 (0.050753)	4.08431 (0.069618)	0.20132 (0.000529)	0.59736 (0.000791)
$\Phi = (\hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3, \hat{p}_1, \hat{p}_2)$					
$\Phi = (2, 3, 4, 0.20, 0.60)$ GP					
n	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{p}_1$	$\hat{p}_2$
50	2.04143 (0.379034)	3.10604 (0.326937)	4.42386 (2.101710)	0.17089 (0.002297)	0.51806 (0.004048)
100	2.04769 (0.200731)	3.05268 (0.156494)	4.26802 (0.943645)	0.18404 (0.001345)	0.55508 (0.002211)
200	2.00909 (0.098997)	3.02822 (0.076810)	4.12159 (0.431607)	0.19162 (0.000732)	0.57640 (0.001153)
300	2.02312 (0.067357)	3.02583 (0.051004)	4.08484 (0.281109)	0.19431 (0.000502)	0.58400 (0.000779)