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ECONOMIC DESIGN OF FRACTION NON-CONFIRMING QUALITY CONTROL CHART

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ABSTRACT

Control charts are widely used to establish and maintain statistical control of a process. They are also effective devices for estimating process parameters, particularly in process- capability studies. The design of a control chart requires the engineer or analyst to select a sample size, a sampling frequency and the control limits for the chart. Selection of these three parameters is usually called the design of the control charts.

Traditionally, control charts have been designed with respect to statistical criteria only. This usually involves selecting the sample size and control limits for predetermined values of α and β risks. The design of a control chart has economic consequences in that the cost of sampling, cost of testing, cost associated with investigating out-of-control signals, cost of correcting assignable causes and cost of allowing non-confirming units to reach the consumer are all affected by the choice of the control chart parameters. Therefore, it is logical to consider the design of a control chart from an economic viewpoint.

This research paper proposed an economic model for Fraction Non-confirming Quality Control Chart (p-chart), by presenting an algorithm for determining the most economic control parameters such as sample size, sampling frequency, and the control limits that will yield maximum average net income of the process. Finally, the output of the economic model was presented in a tabular form which provides the quality controller to choose the desired control chart parameters and maximum income according to α and β risks.

1. INTRODUCTION

It is essential that products meet the requirements of those who use them. Therefore, we can define quality as fitness for use. Quality Control is the set of operational activities that a company uses to ensure that quality characteristics are at the nominal or required levels. Most organizations find it difficult (and expensive) to provide the customer with products that have flawless quality characteristics. A major reason for this difficulty is variability. Since variation can only be describe in statistical terms, statistical methods are to be considered in Quality control effort. The **Quality control chart** is an on-line process control technique widely used for this purpose.

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The design of a control chart has economic consequences in that the cost of sampling, cost of testing, cost associated with investigating out-of-control signals, cost of correcting assignable causes and cost of allowing non-confirming units to reach the consumer are all affected by the choice of the control chart parameters.
In this paper, we present an economic model for Fraction Non-confirming Quality Control Chart (p-chart), by presenting an algorithm for determining the most economic control parameters that will yield maximum average net income of the process.

The rest of this paper is organized as follows. Section 2 presents the literature survey. Section 3 describes the formulation of the problem. Section 4 presents the experimental results. Finally, section 5 concludes the paper.

2. LITERATURE SURVEY

2.1. Various process models in the literature

In 1956, Duncan[1] proposed an economic model for the optimum economic design of the $\bar{x}$ control chart. Duncan drew on the earlier work of Girshick and Rubin[2] in that he utilized a design criterion that maximize the expected net income per unit of time from the process. In the development of the cost model, Duncan assumes that the process is characterized by an in-control state with process mean μ_0 and that a single assignable cause of magnitude δ (measured by standard deviation units), which occurs at random, result in a shift in the mean from μ_0 to either $\mu_0 + \delta\sigma$ or $\mu_0 - \delta\sigma$, where σ is the process standard deviation. The process is monitored by an $\bar{x}$ chart with center line μ_0 and upper and lower control limits $\mu_0 \pm k(\sigma / \sqrt{n})$, where k is the multiple of the σ ,and n is the sample size . Samples are to be taken at intervals of h hours. The parameters μ_0, δ and σ are assumed known ,while n (sample size), k, and h are to be determined. The optimization procedure suggested is based on solving numerical approximations to the system for first partial derivatives of expected loss per hour (*E(L)*) with respect to n, k and h. An iterative procedure is required to solve for the optimal n and k. A close-form solution for h is given using the optimal values of n and k. Several authors have presented optimization methods for Duncan's model. Goel, Jain, and Wu[3] et al. have devised an iterative procedure for minimizing *E(L)* that will produce the exact optimum solution. This paper also contains an extensive sensitivity analysis. Chui and Wetherill[4] have developed a simple, approximate procedure for optimizing Duncan's model.
The development of economic models for the design of control charts has been concentrated on the $\bar{x}$ chart and the control chart for fraction nonconforming. However, the general approach has been extended to other types of control charts. To summarize briefly, the economic design of cumulative-sum control charts was first investigated by Taylor[5]. His model expresses the expected cost per unit of time as a function of the sample size n, sampling interval h, and the V-mask design parameters d and θ. He assumes that a single assignable cause of magnitude δ occurs according to a Poisson process. Goel and Wu[6] have also developed a single assignable- cause model for the optimum economic design of cumulative-sum control charts. Goel[7] has compared economically optimal $\bar{x}$ and cumulative-

sum charts. He reports that, in general, there is a little difference between the two types of charts, with respect to optimum system cost. Chui[8] has also developed a single assignable-cause economic model of the cumulative-sum, following the general modeling strategy of Duncan's[1] $\bar{x}$ chart model.

3. FORMULATION OF THE PROBLEM

3.1. Formulation of the loss-cost function

Economic models are generally formulated using total cost function, which expresses the relationship between the control chart design parameters and three types of costs namely; cost of taking a sample, cost of finding assignable cause, cost of investigating a false alarm, two types of incomes namely; income per hour in the in-control state, income per hour in out-of control state and four periods namely; in-control period, out-of control period, time to take the sample and interpret the result and time to find assignable cause.

Each production cycle is started in the in-control state and continues until process monitors an out-of-control signal. After the necessary adjustments, the process is returned to the in-control state and a new cycle begins. Let *E(T)* be the expected length of a cycle, and *E(C)* be the expected total income during a cycle. Then expected income per unit time is given by,

$$E(I) = \frac{E(C)}{E(T)} \tag{1}$$

The process is assumed to start in an in-control state with a proportion of defective items p_0, where p_0 is a known constant. Because of a single assignable cause of variation, the proportion of defective items has increased to p_1, where p_1 is known constant, and is larger than p_0. The assignable cause occurs at the rate of λ per hour of operating time.

Time interval that the process remains in-control is an exponential random variable with mean $1/\lambda$ hours. Samples of size n are drawn every h hours of production and the process is carried out undisturbed if the number of defectives found in the sample is d or less. The number d is called the *acceptance no.* if the number of defectives found in the sample exceeds d, the process is stopped and a search for an assignable cause is undertaken.

Given the occurrence of the assignable cause between the j^{th} and $(j+1)^{st}$ samples, the expected time of occurrence within this interval is [1],

$$\gamma = \frac{\int_{jh}^{(j+1)h} e^{-\lambda t}\lambda(t - jh)\,dt}{\int_{jh}^{(j+1)h} e^{-\lambda t}\lambda\,dt} = \frac{1-(1+\lambda h)e^{-\lambda h}}{\lambda(1-e^{-\lambda h})} \tag{2}$$

When an assignable cause occurs, the probability that it will be detected on any subsequent sample and the probability of false alarm are,

$$\beta = \sum_{x=0}^{d} \binom{n}{x} p_1^x (1-p_1)^{n-x} \quad , \qquad \alpha = 1 - \sum_{x=0}^{d} \binom{n}{x} p_0^x (1-p_0)^{n-x}$$, respectively.

The time required to take a sample and interpret the result is a constant (*g)* proportional to the sample size, so that *gn* is the length of this segment of the cycle. The time required to find an assignable cause following an out-of control signal is a constant *D*. Therefore, the expected length of a cycle is,

$$E(T) = \frac{1}{\lambda} + \frac{h}{1-\beta} - \gamma + gn + D \quad (3)$$

The net income per hour of operation in the in-control state is V_0, and the net income per hour of operation in the out-of-control state is V_1. The cost of taking a sample of size n is assumed to be of the form $a_1 + a_2 n$; that is , a_1 and a_2 represent, respectively, the fixed and variable components of sampling cost. The expected number of samples taken within a cycle is the expected cycle length divide by the interval between samples, or *E(T)/h.*
The cost of finding an assignable cause is a_3, and the cost of investigating a false alarm is a_3'. The expected number of false alarms generated during a cycle is α times the expected number of samples taken before the shift,

$$\alpha \times \left(\sum_{j=0}^{\infty} \int_{jh}^{(j+1)h} je^{-\lambda t} dt \right) = \frac{\alpha e^{-\lambda h}}{1 - e^{-\lambda h}} \qquad (4)$$

Therefore, the expected net income per cycle is,

$$E(C) = V_0 \frac{1}{\lambda} + V_1 \left(\frac{h}{1-\beta} - \gamma + gn + D \right) - a_3 - \frac{a_3' e^{-\lambda h}}{1-e^{-\lambda h}} - (a_1 + a_2 n) \frac{E(T)}{h} \qquad (5)$$

The expected net income per hour is found by dividing the expected net income per cycle, equation (5), by the expected cycle length, equation (3), resulting in

$$E(I) = V_0 - \frac{a_1 + a_2 n}{h} - \frac{a_4[h/(1-\beta) - \gamma + gn + D] + a_3 + a_3' \alpha e^{-\lambda h} / (1-e^{-\lambda h})}{1/\lambda + h/(1-\beta) - \gamma + gn + D} \qquad (6)$$

Let $a_4 = V_0 - V_1$ be the hourly penalty cost associated with production in the out-of-control state. Then equation (6) may be rewritten as

$$E(L) = \frac{a_1 + a_2 n}{h} + \frac{a_4[h/(1-\beta) - \gamma + gn + D] + a_3 + a_3' \alpha e^{-\lambda h} / (1-e^{-\lambda h})}{1/\lambda + h/(1-\beta) - \gamma + gn + D} \qquad (7)$$

The expression *E(L)* represents the expected loss per hour incurred by the process. *E(L)* is a function of the control chart parameters *n, k,* and *h.* That is clearly, maximizing the expected net income per hour is equivalent to minimizing *E(L).*

3.2. *Optimisation Procedure*

Since the optimization procedure was started for predefined sample size, ultimately optimal values for the sampling frequency (*h*) and control limit (*k*) and α, β were obtained. Then optimization was carried out until it gives the Best Simulation for minimum Expected loss *E(L)*. This procedure was carried out for fixed shift (δ) that is keeping $p_1 - p_0$ constant and for the sample sizes starting from $n = 10$ to $n = 110$ and obtain the minimum loss cost *E(L)* for each sample size.

Then by varying the shift same procedure was carried out for five different shifts ($\delta = 0.1, \delta = 0.5, \delta = 1.0, \delta = 1.5, \delta = 2.0$) and the minimum loss cost *E(L)* was obtained. Finally, the output was presented in a tabular form which provides the quality controller to choose the desired control chart parameters and maximum income according to α and β risks.

4. RESULTS

Table 1. Results for Shift Level = 0.1

n	10	15	20	25	30	35	40	45	50	55	60
h	1	1	1	1	1	1	1	1	1	1	1
α	0.196	0.140	0.162	0.222	0.026	0.030	0.033	0.036	0.040	0.042	0.045
β	0.497	0.439	0.368	0.304	0.246	0.193	0.146	0.003	0.064	0.029	0.098
k	3.765	3.592	3.272	2.543	3.000	3.447	2.053	2.000	3.631	3.656	3.679
E(L)	7.083	8.075	8.661	10.524	11.312	11.449	12.040	12.294	13.547	13.763	14.943
E(I)	492.917	491.926	491.339	489.476	488.688	488.551	487.960	487.706	486.453	486.237	485.057

n	65	70	75	80	85	90	95	100	105	110
h	1	1	1	1	1	1	1	1	1	1
α	0.048	0.051	0.053	0.055	0.020	0.023	0.025	0.026	0.029	0.031
β	0.069	0.043	0.020	0.099	0.049	0.046	0.043	0.040	0.038	0.035
k	3.683	4.456	4.326	4.130	4.331	4.495	4.593	4.629	4.654	4.655
E(L)	15.092	16.204	17.286	18.339	15.927	16.870	17.806	18.752	19.699	20.644
E(I)	484.908	483.796	482.714	481.661	484.073	483.130	482.194	481.248	480.301	479.356

Table 2 Results for Shift Level = 0.5

n	10	15	20	25	30	35	40	45	50	55	60
h	1	1	1	1	1	1	1	1	1	1	1
α	0.096	0.140	0.182	0.222	0.036	0.048	0.061	0.010	0.014	0.018	0.022
β	0.439	0.395	0.290	0.213	0.156	0.130	0.099	0.049	0.042	0.035	0.029
k	3.702	3.601	3.251	2.000	3.250	3.846	3.972	4.041	4.904	4.412	3.175
E(L)	8.435	9.764	11.283	12.845	13.614	14.296	14.381	14.480	14.869	15.534	16.253
E(I)	491.565	490.236	488.717	487.155	486.386	485.704	485.619	485.520	485.131	484.466	483.747

n	65	70	75	80	85	90	95	100	105	110
h	1	1	1	1	1	1	1	1	1	1
α	0.004	0.005	0.007	0.087	0.011	0.013	0.003	0.003	0.004	0.005
β	0.045	0.039	0.033	0.029	0.024	0.020	0.032	0.028	0.024	0.020
k	4.645	4.689	4.428	3.752	3.861	3.339	4.187	4.813	4.204	4.137
E(L)	16.891	17.487	18.124	18.794	19.488	20.204	21.865	22.521	23.194	23.880
E(I)	483.109	482.514	481.876	481.206	480.512	479.796	478.135	477.479	476.806	476.120

Table 3 Results for Shift Level = 1.0

n	10	15	20	25	30	35	40	45	50	55	60
h	1	1	1	1	1	1	1	1	1	1	1
α	0.096	0.010	0.017	0.026	0.003	0.005	0.008	0.010	0.014	0.002	0.003
β	0.312	0.497	0.338	0.222	0.344	0.244	0.169	0.115	0.076	0.132	0.092
k	3.000	4.852	4.134	3.665	4.990	4.583	4.357	4.395	3.952	4.744	4.442
E(L)	11.008	11.013	11.258	12.020	12.878	13.509	14.303	15.192	16.142	16.943	17.821
E(I)	488.992	488.987	488.742	487.981	487.123	486.491	485.697	484.808	483.858	483.057	482.179

n	65	70	75	80	85	90	95	100	105	110
h	1	1	1	1	1	1	1	1	1	1
α	0.042	0.054	0.001	0.001	0.017	0.002	0.003	0.003	0.001	0.000
β	0.063	0.043	0.074	0.052	0.036	0.025	0.017	0.011	0.021	0.015
k	4.338	4.253	4.999	4.812	4.717	4.352	4.287	4.936	4.860	4.775
E(L)	18.732	19.667	20.567	21.471	22.389	23.316	24.250	25.190	26.074	26.996
E(I)	481.268	480.333	479.433	478.529	477.611	476.684	475.750	474.811	473.926	473.004

Table 4 Results for Shift Level = 1.5

n	10	15	20	25	30	35	40	45	50	55	60
h	1	1	1	1	1	1	1	1	1	1	1
α	0.095	0.010	0.017	0.026	0.003	0.005	0.008	0.010	0.002	0.002	0.003
β	0.175	0.282	0.147	0.074	0.120	0.066	0.035	0.018	0.031	0.017	0.009
k	3.000	4.919	4.110	3.905	4.962	4.817	4.703	4.219	4.999	4.796	4.668
E(L)	13.795	13.419	14.176	15.327	16.139	17.188	18.337	19.542	20.552	21.705	22.878
E(I)	486.205	486.581	485.824	484.673	483.861	482.812	481.663	480.458	479.449	478.295	477.122

n	65	70	75	80	85	90	95	100	105	110
h	1	1	1	1	1	1	1	1	1	1
α	0.004	0.005	0.001	0.001	0.002	0.002	0.003	0.003	0.001	0.001
β	0.005	0.002	0.005	0.002	0.001	0.001	0.000	0.000	0.000	0.000
k	4.245	4.860	4.959	4.924	4.616	4.379	4.472	4.848	4.926	4.842
E(L)	24.063	25.255	26.321	27.485	28.651	29.816	30.982	32.148	33.231	34.380
E(I)	475.937	474.745	473.679	472.515	471.350	470.184	469.018	467.853	466.769	465.621

Table 5 Results for Shift Level = 2.0

n	10	15	20	25	30	35	40	45	50	55	60
h	1	1	1	1	1	1	1	1	1	1	1
α	0.096	0.010	0.017	0.026	0.003	0.005	0.008	0.010	0.002	0.002	0.003
β	0.095	0.075	0.057	0.021	0.034	0.014	0.005	0.002	0.004	0.001	0.001
k	3.324	4.893	4.392	4.813	4.954	4.895	4.452	4.899	4.982	4.756	4.811
E(L)	16.507	16.704	17.301	18.801	19.736	21.124	22.566	24.035	25.240	26.652	28.070
E(I)	483.493	483.296	482.699	481.199	480.264	478.876	477.434	475.965	474.760	473.348	471.930
n	65	70	75	80	85	90	95	100	105	110	
h	1	1	1	1	1	1	1	1	1	1	
α	0.004	0.005	0.001	0.001	0.002	0.002	0.003	0.003	0.001	0.001	
β	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	
k	4.379	4.708	4.951	4.858	4.766	4.685	4.656	4.433	4.889	4.715	
E(L)	29.489	30.910	32.192	33.583	34.971	36.358	37.742	39.125	40.424	41.787	
E(I)	470.511	469.090	467.808	466.417	465.029	463.643	462.258	460.875	459.576	458.213	

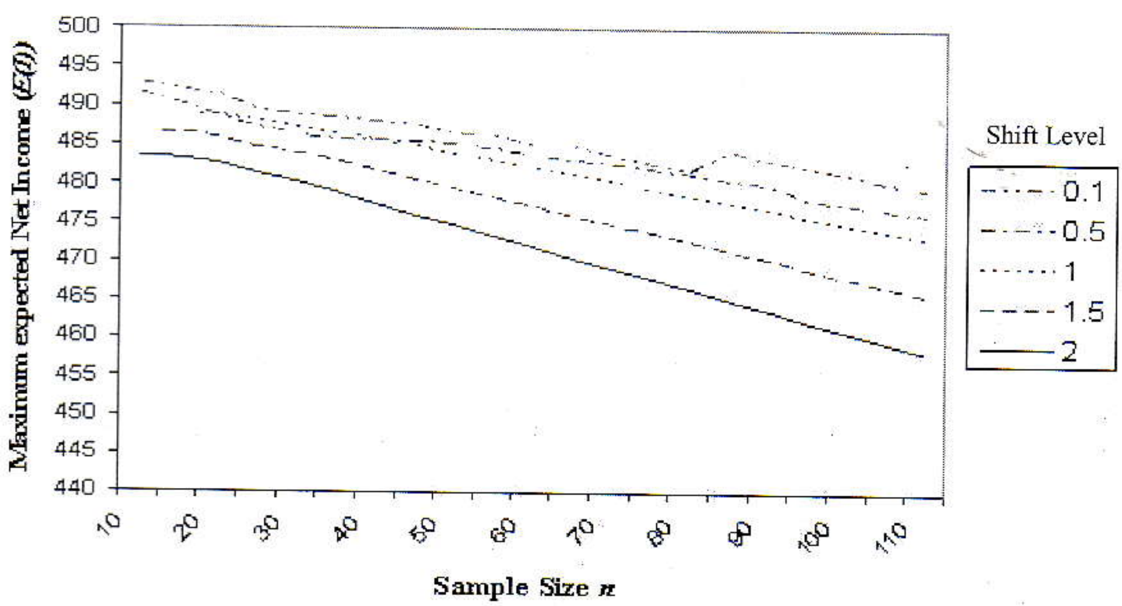

Fig. 1 Graph of Maximum Expected Net Income Vs Sample Size

5. CONCLUSION

In general, it was found that smaller shifts require larger samples and for relatively larger shifts the maximum net income is relatively small. Also when the sample size (n) increases maximum expected net income ($E(I)$) decreases. This is clearly shown in Fig. 1.

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