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RESEARCH ARTICLE

Ordered Random Variables

On recurrence relations for moments of dual generalized order statistics for a general transmuted power function distributions with characterizations

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Abstract: Dual generalized order statistics is a unified method for random variables that are arranged in decreasing order. The moments of dual generalized order statistics are helpful to study the properties of any distribution. Often, the moments of dual generalized order statistics are not easy to compute and recursive computation is done. The recurrence relations for moments of generalized and dual generalized order statistics are helpful to compute the higher order moments from the lower order moments. In this paper the methods for recursive computation of moments of dual generalized order statistics for general transmuted power function distributions are presented. The general transmuted power function distributions are first defined and then the recurrence relations are obtained. These recurrence relations include relations for single, inverse, product, and ratio moments. The recurrence relations are used to obtain the relations for moments of special cases, which include lower record values and reversed order statistics. Some characterizations of the general transmuted power function distributions are also presented based on the basis of single and product moments of dual generalized order statistics. These characterizations are unique results for the general transmuted power function distributions. The results given in the paper are useful to obtain the results for special cases of general transmuted power function distribution which includes power function and transmuted power function distributions.

Keywords: Dual generalized order statistics, general transmuted distributions, moments, power function distribution.

INTRODUCTION

The study of ordered random variables has been an area of interest for several authors. The classical method for increasingly arranged random variables is order statistics and has been a very useful method in studies of extreme events. This method has been extensively discussed by various authors; see for example David and Nagaraja (2003) and Arnold *et al.* (2008). Chandler (1952) and Dziubdziel and Kopociski (1976) have introduced record values as another method for the modelling of extreme records – lower as well as upper records.

Kamps (1995) has introduced generalized order statistics (gos) as a unified model for random variables which are arranged in increasing order. Kamps (1995) has argued that several models for increasingly arranged random variables, including order statistics and upper record values, appear as special cases of the generalized order statistics.

In several situations we need to study the distributional behaviour of random variables which are arranged in decreasing order of importance. Reflected order statistics and lower record values are two classical methods for such random variables. The dual generalized order statistics (dgos), introduced by Burkschat *et al.* (2003), is a unified method for random variables arranged in decreasing order.

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Burkschat *et al.* (2003) have shown that the joint distribution of *dgos* for a sample of size n is

$$f_{1,2,\ldots,n:n,m,k}(x_1,\ldots,x_n)=k\left(\prod_{j=1}^{n-1}\gamma_j\right)[F(x_n)]^{k-1}f(x_n)\left[\prod_{i=1}^{n-1}\{F(x_i)\}^m f(x_i)\right],x\in\Re, \qquad \ldots(1)$$

where $k\geq 1$, $m\in$, $\gamma_j=k+(n-j)(m+1)$ and $F(x_i)$ is cumulative distribution function of i^{th} random variable. Burkschat *et al.* (2003) have further shown that the marginal distribution of a single *dgos* and joint distribution of two *dgos* are given as

$$f_{r:n,m,k}(x)=\frac{C_{r-1}}{(r-1)!}f(x)\left[F(x)\right]^{\gamma_r-1}g_m^{r-1}\left[F(x)\right],\ x\in\Re\ , \qquad \ldots(2)$$

and

$$f_{r,s:n,m,k}(x_1,x_2)=\frac{C_{s-1}}{(r-1)!(s-r-1)!}f(x_1)f(x_2)\left[F(x_1)\right]^m g_m^{r-1}\left[F(x_1)\right]$$
$$\times\left[F(x_2)\right]^{\gamma_s-1}\left[h_m\{F(x_1)\}-h_m\{F(x_2)\}\right]^{s-r-1},x\in\Re \qquad \ldots(3)$$

where $C_{r-1}=\prod_{j=1}^{r}\gamma_j$ and

$$h_m(x)=\begin{cases}(m+1)^{-1}\left(x^{m+1}\right)\ ;\ m\neq-1\\ \ln x; \qquad m=-1.\end{cases}\ ;\ g_m(x)=\begin{cases}(m+1)^{-1}\left(1-x^{m+1}\right)\ ;\ m\neq-1\\ -\ln x; \qquad m=-1.\end{cases}$$

Different models of decreasingly arranged random variables appear as special cases of *dgos*. For example, the reflected order statistics appears as a special case of *dgos* for $m = 0$ and $k = 1$. The lower record values appear as special case of *dgos* for $m = –1$. For further discussion about *dgos* see, for example, Ahsanullah and Nevzorov (2001) and Shahbaz *et al.* (2016)

Most of the work in the field of *dgos* is based upon obtaining recurrence relations for moments of *dgos,* when the sample is available from some probability distribution, and proposing some characterizations of the distributions using moments of *dgos*. Athar and Faizan (2011) have obtained the relations for moments of lower record values for power function distribution and have also obtained some characterizations. The recurrence relations for moments of *dgos* for a general class of inverted distributions have been obtained by Kotb *et al.* (2013). Ahsanullah *et al.* (2013) have given some characterizations for the power function distribution using moments of lower record values.

The recurrence relations for moments of *gos* and *dgos* for transmuted distributions have not been explored. Al-Sobhi *et al.* (2021) have obtained the recurrence relations for moments of *gos* for a transmuted exponential distribution. In this paper we have obtained the recurrence relations for moments of *dgos* for a general transmuted power function distribution. We will first introduce the general transmuted power function distribution.

MATERIALS AND METHODS

In this section the general transmuted power function distributions have been introduced.

General transmuted power function distributions

The transmuted family of distributions has been introduced by Shaw and Buckley (2007) as a method of obtaining new probability distributions from the available ones. This family has been explored by several authors. Recently, Rehman *et al.* (2018; 2018b) have extended the transmuted family of distributions to the general case and have proposed two methods to extend them. The distribution functions of the general transmuted family of distributions, proposed by Rehman *et al.* (2018a; 2018b), are

$$F_1(x)=G(x)+\sum_{h=1}^{t}\lambda_h G^h(x)\left[1-G(x)\right], x\in\Re \qquad \ldots(4)$$

and

$$F_2(x)=G(x)+\sum_{h=1}^{t}\lambda_h G(x)\left[1-G(x)\right]^h, x\in\Re\,, \qquad \ldots(5)$$

where $G(x)$ is cumulative distribution function (*cdf*) of the baseline distribution. The density functions corresponding to (4) and (5) are, respectively,

$$f_1(x)=g(x)\left[1-\sum_{h=1}^{t}(h+1)\lambda_h G^h(x)+\sum_{h=1}^{t}h\lambda_h G^{h-1}(x)\right], x\in\Re \qquad \ldots(6)$$

and

$$f_2(x)=g(x)\left[1+\sum_{h=1}^{t}\lambda_h\left\{1-G(x)\right\}^h-\sum_{h=1}^{t}h\lambda_h G(x)\left\{1-G(x)\right\}^{h-1}\right], x\in\Re\,, \qquad \ldots(7)$$

where $g(x)$ is the density function corresponding to $G(x)$. Also, the $\lambda's$ are the transmutation parameters. These parameters require certain conditions that are given below.

For (4) and (6): $-1\le\lambda_h\le 1 \;\&\; -t\le\sum_{h=1}^{t}\lambda_h\le 1$

For (5) and (7): $-1\le\lambda_h\le 1 \;\&\; -1\le\sum_{h=1}^{t}\lambda_h\le t$.

The transmuted family of distributions, proposed by Shaw and Buckley (2007), appears as a special case of the general transmuted family of distributions for $t = 1$. The general transmuted family of distributions can be used to generate transmuted distributions for different $G(x)$. In the following we have proposed two general transmuted power function distributions by using the following density and distribution functions of power function distribution, with parameters $(\alpha,c)>0$,

$$g(x)=\frac{\alpha x^{\alpha-1}}{c^\alpha}, 0<x<c, (\alpha,c)>0 \;;\; G(x)=\frac{x^\alpha}{c^\alpha}, 0<x<c, (\alpha,c)>0\,. \qquad \ldots(8)$$

in (4) and (5). These distributions are defined in the following subsections.

The general transmuted power function distribution–I

The general transmuted power function distribution-I (GTPF-I) is obtained by using (8) in (4) and (6). The distribution and density function of the *GTPF-I* are

$$F_1(x)=\frac{x^\alpha}{c^\alpha}+\sum_{h=1}^{t}\lambda_h\frac{x^{\alpha h}}{c^{\alpha h}}\left(1-\frac{x^\alpha}{c^\alpha}\right)=\frac{x^\alpha}{c^\alpha}\left[1+\sum_{h=1}^{t}\lambda_h\frac{x^{\alpha(h-1)}}{c^{\alpha(h-1)}}-\sum_{h=1}^{t}\lambda_m\frac{x^{\alpha h}}{c^{\alpha h}}\right], 0<x<c \qquad \ldots(9)$$

and

$$f_1(x)=\frac{\alpha x^{\alpha-1}}{c^\alpha}\left[1+\sum_{h=1}^{t}h\lambda_h\frac{x^{\alpha(h-1)}}{c^{\alpha(h-1)}}-\sum_{h=1}^{t}(h+1)\lambda_h\frac{x^{\alpha h}}{c^{\alpha h}}\right], 0<x<c\,. \qquad \ldots(10)$$

It is to be noted that the transmuted power function distribution, proposed by Haq *et al.* (2016), appears as a special case of *GTPF–I* for $t = 1$. The density and distribution functions of *GTPF-I* are related as

$$F_1(x)=\frac{x}{\alpha}f_1(x)-\sum_{h=1}^{t}(h-1)\lambda_h\frac{x^{\alpha h}}{c^{\alpha h}}+\sum_{h=1}^{t}h\lambda_h\frac{x^{\alpha(h+1)}}{c^{\alpha(h+1)}}, 0<x<c\,. \qquad \ldots(11)$$

This relation is useful in obtaining recurrence relations for moments of *dgos* for *GTPF-I* which we will give in the coming sections.

The general transmuted power function distribution–II

The general transmuted power function distribution–II (GTPF–II) is obtained by using (8) in (5) and (7). The distribution and density function of the *GTPF–II* are

$$F_2(x)=\frac{x^\alpha}{c^\alpha}+\sum_{h=1}^{t}\lambda_h\frac{x^\alpha}{c^\alpha}\left(1-\frac{x^\alpha}{c^\alpha}\right)^n,\ 0<x<c \quad \ldots(12)$$

and

$$f_2(x)=\frac{\alpha x^{\alpha-1}}{c^\alpha}\left[1+\sum_{h=1}^{t}\lambda_h\left(1-\frac{x^\alpha}{c^\alpha}\right)^h-\sum_{h=1}^{t}h\lambda_h\frac{x^\alpha}{c^\alpha}\left(1-\frac{x^\alpha}{c^\alpha}\right)^{h-1}\right],\ 0<x<c\,. \quad \ldots(13)$$

The distribution and density functions, given in (12) and (13) are related as

$$F_2(x)=\frac{x}{\alpha}f_2(x)+\sum_{h=1}^{t}h\lambda_h\sum_{j=0}^{h-1}(-1)^j\binom{h-1}{j}\frac{x^{\alpha(j+2)}}{c^{\alpha(j+2)}},\ 0<x<c\ . \quad \ldots(14)$$

The relation (14) is useful in obtaining recurrence relations for moments of *dgos* for *GRPF–II* which will be obtained in the coming sections.

RESULTS AND DISCUSSION

In this section the main results of the paper are discussed.

Recurrence relations for single and inverse moments

In this section we will obtain the recurrence relations for single and inverse moments of *dgos* for the general transmuted power function distributions, introduced in the previous section. For this we will consider following relation between moments of *dgos*, see for example Kotb *et al.* (2003) and Shahbaz *et al.* (2016).

$$\mu_{r:n,m,k}^{p}-\mu_{r-1:n,m,k}^{p}=-\frac{pC_{r-1}}{\gamma_r(r-1)!}\int_{-\infty}^{\infty}x^{p-1}\left[F(x)\right]^{\gamma_r}g_m^{r-1}\left[F(x)\right]dx\,, \quad \ldots(15)$$

where $\mu_{r:n,m,k}^{p}=E\left(X_{r:n,m,k}^{p}\right)$ and $X_{r:n,m,k}$ is the r^{th} *dgos*.

The recurrence relations for general transmuted power function distributions are obtained in the following subsections.

General transmuted power function distribution–I

In this section we will obtain the recurrence relations for single and inverse moments of *dgos* for *GTPF-I*. The recurrence relation for single moments of *dgos* from *GTPF–I* is given in the following Theorem.

Theorem 1: The single moments of general transmuted power function distribution-I are related as

$$\mu_{r:n,m,k}^{p}=\left(\frac{\alpha\gamma_r}{\alpha\gamma_r+p}\right)\left[\mu_{r-1:n,m,k}^{p}+\sum_{h=1}^{t}\frac{\lambda_h p\gamma_{r(k-1)}C_{r-1}}{c^{\alpha(h+1)}\gamma_r C_{r-1(k-1)}}\left\{\frac{h}{p+\alpha(h+1)}\right.\right.$$
$$\left.\left.\times\left[\mu_{r:n,m,k-1}^{p+\alpha(h+1)}-\mu_{r-1:n,m,k-1}^{p+\alpha(h+1)}\right]-\frac{c^\alpha(h-1)}{p+\alpha h}\left[\mu_{r:n,m,k-1}^{p+\alpha h}-\mu_{r-1:n,m,k-1}^{p+\alpha h}\right]\right\}\right], \quad \ldots(16)$$

where $\gamma_{r(k-1)} = (k-1)+(n-r)(m+1)$ and $C_{r-1(k-1)} = \prod_{j=1}^{r} \gamma_{j(k-1)}$.

Proof: Using (11) in (15) we have

$$\mu_{r:n,m,k}^{p} - \mu_{r-1:n,m,k}^{p} = -\frac{pC_{r-1}}{\gamma_r (r-1)!} \int_0^c x^{p-1} \left\{ \frac{x}{\alpha} f_1(x) - \sum_{h=1}^{t} (h-1)\, \lambda_h \frac{x^{\alpha h}}{c^{\alpha h}} + \sum_{h=1}^{t} h\, \lambda_h \frac{x^{\alpha(h+1)}}{c^{\alpha(h+1)}} \right\}$$

$$\times \left[F(x)\right]^{\gamma_r - 1} g_m^{r-1}\left[F(x)\right] dx$$

$$= -\frac{p}{\alpha\gamma_r} \mu_{r:n,m,k}^{p} + \sum_{h=1}^{t} (h-1)\, \lambda_h \frac{pC_{r-1}}{\gamma_r c^{\alpha h} (r-1)!} \int_0^c x^{p+\alpha h-1} \left[F(x)\right]^{\gamma_r - 1} g_m^{r-1}\left[F(x)\right] dx$$

$$- \sum_{h=1}^{t} h\lambda_h \frac{pC_{r-1}}{\gamma_r c^{\alpha(h+1)} (r-1)!} \int_0^c x^{p+\alpha(h+1)-1} \left[F(x)\right]^{\gamma_r - 1} g_m^{r-1}\left[F(x)\right] dx$$

$$= -\frac{p}{\alpha\gamma_r} \mu_{r:n,m,k}^{p} - \sum_{h=1}^{t} (h-1) \frac{\lambda_h p \gamma_{r(k-1)} C_{r-1}}{\gamma_r c^{\alpha h} (p+\alpha h) C_{r-1(k-1)}} \left\{ \mu_{r:n,m,k-1}^{p+\alpha h} - \mu_{r-1:n,m,k-1}^{p+\alpha h} \right\}$$

$$+ \sum_{h=1}^{t} h \frac{\lambda_h p \gamma_{r(k-1)} C_{r-1}}{\gamma_r c^{\alpha(h+1)} \left\{p+\alpha(h+1)\right\} C_{r-1(k-1)}} \left\{ \mu_{r:n,m,k-1}^{p+\alpha(h+1)} - \mu_{r-1:n,m,k-1}^{p+\alpha(h+1)} \right\}.$$

Rearranging, we have (16) and hence the theorem.

It is to be noted that the recurrence relation (16) reduces to the recurrence relations for moments of *dgos* for power function distribution for $\lambda_h = 0;\ h = 1,2,\ldots,t$ as obtained by Athar and Faizan (2011). The following corollaries immediately follow from Theorem 1.

Corollary 1: Using "–*p*" in place of *p* the following relation for inverse moments of *dgos* for *GTPF-I* is obtained

$$\mu_{r:n,m,k}^{-p} = \left(\frac{\alpha\gamma_r}{\alpha\gamma_r - p}\right) \left[\mu_{r-1:n,m,k}^{-p} - \sum_{h=1}^{t} \frac{\lambda_h p \gamma_{r(k-1)} C_{r-1}}{c^{\alpha(h+1)} \gamma_r C_{r-1(k-1)}} \left\{ \frac{h}{\alpha(h+1)-p} \right. \right.$$

$$\left.\left. \times \left[\mu_{r:n,m,k-1}^{\alpha(h+1)-p} - \mu_{r-1:n,m,k-1}^{\alpha(h+1)-p} \right] - \frac{c^{\alpha}(h-1)}{\alpha h - p} \left[\mu_{r:n,m,k-1}^{\alpha h-p} - \mu_{r-1:n,m,k-1}^{\alpha h-p} \right] \right\} \right]. \quad \ldots (17)$$

Corollary 2: Using $m = -1$ in (16) and (17) the following relations for single and inverse moments of lower record values for *GTPF-I* are obtained

$$\mu_{K(r)}^{p} = \left(\frac{\alpha k}{\alpha k + p}\right) \left[\mu_{K(r-1)}^{p} + \sum_{h=1}^{t} \frac{\lambda_h p}{c^{\alpha(h+1)}} \left(\frac{k}{k-1}\right)^{r-1} \left\{ \frac{h}{p+\alpha(h+1)} \right. \right.$$

$$\left.\left. \times \left[\mu_{K-1(r)}^{p+\alpha(h+1)} - \mu_{K-1(r-1)}^{p+\alpha(h+1)} \right] - \frac{c^{\alpha}(h-1)}{p+\alpha h} \left[\mu_{K-1(r)}^{p+\alpha h} - \mu_{K-1(r-1)}^{p+\alpha h} \right] \right\} \right]. \quad \ldots (18)$$

$$\mu_{K(r)}^{-p} = \left(\frac{\alpha k}{\alpha k - p}\right) \left[\mu_{K(r-1)}^{-p} - \sum_{h=1}^{t} \frac{\lambda_h p}{c^{\alpha(h+1)}} \left(\frac{k}{k-1}\right)^{r-1} \left\{ \frac{h}{\alpha(h+1)-p} \right. \right.$$

$$\left.\left. \times \left[\mu_{K-1(r)}^{\alpha(h+1)-p} - \mu_{K-1(r-1)}^{\alpha(h+1)-p} \right] - \frac{c^{\alpha}(h-1)}{\alpha h - p} \left[\mu_{K-1(r)}^{\alpha h-p} - \mu_{K-1(r-1)}^{\alpha h-p} \right] \right\} \right]. \quad \ldots (19)$$

Corollary 3: Using m = 0 and k = 1 in (16) and (17) the following relations for single and inverse moments of reversed order statistics for GTPF-I are obtained.

$$\mu_{r:n}^{p}=\left[\frac{\alpha\left(n-r+1\right)}{\alpha\left(n-r+1\right)+p}\right]\left[\mu_{r-1:n}^{p}+\sum_{h=1}^{t}\frac{\lambda_h p}{c^{\alpha(h+1)}}\frac{n}{n-r+1}\left\{\frac{h}{p+\alpha\left(h+1\right)}\right.\right.$$
$$\left.\left.\times\left[\mu_{r:n}^{p+\alpha(h+1)}-\mu_{r-1:n}^{p+\alpha(h+1)}\right]-\frac{c^{\alpha}\left(h-1\right)}{p+\alpha h}\left[\mu_{r:n}^{p+\alpha h}-\mu_{r-1:n}^{p+\alpha h}\right]\right\}\right]. \quad \ldots(20)$$

$$\mu_{r:n}^{-p}=\left[\frac{\alpha\left(n-r+1\right)}{\alpha\left(n-r+1\right)-p}\right]\left[\mu_{r-1:n}^{-p}-\sum_{h=1}^{t}\frac{\lambda_h p}{c^{\alpha(h+1)}}\frac{n}{n-r+1}\left\{\frac{h}{\alpha\left(h+1\right)-p}\right.\right.$$
$$\left.\left.\times\left[\mu_{r:n}^{\alpha(h+1)-p}-\mu_{r-1:n}^{\alpha(h+1)-p}\right]-\frac{c^{\alpha}\left(h-1\right)}{\alpha h-p}\left[\mu_{r:n}^{\alpha h-p}-\mu_{r-1:n}^{\alpha h-p}\right]\right\}\right]. \quad \ldots(21)$$

General transmuted power function distribution-II

In the following we will obtain the recurrence relations for single and inverse moments of *dgos* for *GTPF-II*. The recurrence relation for single moments of *dgos* from *GTPF-II* is given in the following Theorem.

Theorem 2: The single moments of general transmuted power function distribution-II are related as

$$\mu_{r:n,m,k}^{p}=\left(\frac{\alpha\gamma_r}{\alpha\gamma_r+p}\right)\left[\mu_{r-1:n,m,k}^{p}+\sum_{h=1}^{t}h\lambda_h\sum_{j=0}^{h-1}\frac{(-1)^j}{c^{\alpha(j+2)}}\binom{h-1}{j}\right.$$
$$\left.\times\frac{p\gamma_{r(k-1)}C_{r-1}}{\left\{p+\alpha\left(j+2\right)\right\}\gamma_r C_{r-1(k-1)}}\left\{\mu_{r:n,m,k-1}^{p+\alpha(j+2)}-\mu_{r-1:n,m,k-1}^{p+\alpha(j+2)}\right\}\right]. \quad \ldots(22)$$

Proof: The recurrence relation for single moments of *dgos* for *GTPF–II* is obtained by using (14) in (15) as

$$\mu_{r:n,m,k}^{p}-\mu_{r-1:n,m,k}^{p}=-\frac{pC_{r-1}}{\gamma_r\left(r-1\right)!}\int_0^c x^{p-1}\left\{\frac{x}{\alpha}f_2\left(x\right)+\sum_{h=1}^{t}h\lambda_h\sum_{j=0}^{h-1}\left(-1\right)^j\binom{h-1}{j}\frac{x^{\alpha(j+2)}}{c^{\alpha(j+2)}}\right\}$$
$$\times\left[F\left(x\right)\right]^{\gamma_r-1}g_m^{r-1}\left[F\left(x\right)\right]dx$$
$$=-\frac{pC_{r-1}}{\alpha\gamma_r\left(r-1\right)!}\int_0^c x^p f_2\left(x\right)\left[F\left(x\right)\right]^{\gamma_r-1}g_m^{r-1}\left[F\left(x\right)\right]dx-\frac{pC_{r-1}}{\gamma_r\left(r-1\right)!}$$
$$\times\sum_{h=1}^{t}h\lambda_h\sum_{j=0}^{h-1}\frac{(-1)^j}{c^{\alpha(j+2)}}\binom{h-1}{j}\int_0^c x^{p+\alpha(j+2)-1}\left[F\left(x\right)\right]^{\gamma_r-1}g_m^{r-1}\left[F\left(x\right)\right]dx$$
$$=-\frac{p}{\alpha\gamma_r}\mu_{r:n,m,k}^{p}+\sum_{h=1}^{t}h\lambda_h\sum_{j=0}^{h-1}\frac{(-1)^j}{c^{\alpha(j+2)}}\binom{h-1}{j}\frac{p\gamma_{r(k-1)}C_{r-1}}{\left\{p+\alpha\left(j+2\right)\right\}\gamma_r C_{r-1(k-1)}}$$
$$\times\left[\mu_{r:n,m,k-1}^{p+\alpha(j+2)}-\mu_{r-1:n,m,k-1}^{p+\alpha(j+2)}\right].$$

Rearranging the above we have (22) and hence the Theorem.

The relation (22) reduces to the recurrence relations for single moments of *dgos* for power function distribution for $\lambda_h=0$; $h=1,2,\ldots,t$. The following are some corollaries which immediately follow from Theorem 2.

Corollary 4: Using "$-p$" in place of p the following relation for inverse moments of *dgos* for *GTPF-II* is obtained

$$\mu_{r:n,m,k}^{-p}=\left(\frac{\alpha\gamma_r}{\alpha\gamma_r-p}\right)\left[\mu_{r-1:n,m,k}^{-p}-\sum_{h=1}^{t}h\lambda_h\sum_{j=0}^{h-1}\frac{(-1)^j}{c^{\alpha(j+2)}}\binom{h-1}{j}\right.$$
$$\left.\times\frac{p\gamma_{r(k-1)}C_{r-1}}{\{\alpha(j+2)-p\}\gamma_r C_{r-1(k-1)}}\left\{\mu_{r:n,m,k-1}^{\alpha(j+2)-p}-\mu_{r-1:n,m,k-1}^{\alpha(j+2)-p}\right\}\right]. \quad \ldots(23)$$

Corollary 5: Using $m=-1$ in (22) and (23) the following relations for single and inverse moments of lower record values for *GTPF–II* are obtained

$$\mu_{K(r)}^{p}=\left(\frac{\alpha k}{\alpha k+p}\right)\left[\mu_{K(r-1)}^{p}+\sum_{h=1}^{t}h\lambda_h\sum_{j=0}^{h-1}\frac{(-1)^j p}{c^{\alpha(j+2)}\{p+\alpha(j+2)\}}\binom{h-1}{j}\right.$$
$$\left.\times\left(\frac{k}{k-1}\right)^{r-1}\left\{\mu_{K-1(r)}^{p+\alpha(j+2)}-\mu_{K-1(r-1)}^{p+\alpha(j+2)}\right\}\right]. \quad \ldots(24)$$

$$\mu_{K(r)}^{-p}=\left(\frac{\alpha k}{\alpha k-p}\right)\left[\mu_{K(r-1)}^{p}-\sum_{h=1}^{t}h\lambda_h\sum_{j=0}^{h-1}\frac{(-1)^j p}{c^{\alpha(j+2)}\{\alpha(j+2)-p\}}\binom{h-1}{j}\right.$$
$$\left.\times\left(\frac{k}{k-1}\right)^{r-1}\left\{\mu_{K-1(r)}^{\alpha(j+2)-p}-\mu_{K-1(r-1)}^{\alpha(j+2)-p}\right\}\right]. \quad \ldots(25)$$

Corollary 6: Using $m=0$ and $k=1$ in (22) and (23) the following relations for single and inverse moments of reversed order statistics for *GTPF-I* are obtained

$$\mu_{r:n}^{p}=\left[\frac{\alpha(n-r+1)}{\alpha(n-r+1)+p}\right]\left[\mu_{r-1:\mathrm{n}}^{p}+\sum_{h=1}^{t}h\lambda_h\sum_{j=0}^{h-1}\frac{(-1)^j}{c^{\alpha(j+2)}}\binom{h-1}{j}\right.$$
$$\left.\times\frac{p}{\{p+\alpha(j+2)\}}\frac{n}{n-r+1}\left\{\mu_{r:n}^{p+\alpha(j+2)}-\mu_{r-1:n}^{p+\alpha(j+2)}\right\}\right]. \quad \ldots(26)$$

$$\mu_{r:n}^{-p}=\left[\frac{\alpha(n-r+1)}{\alpha(n-r+1)-p}\right]\left[\mu_{r-1:\mathrm{n}}^{-p}-\sum_{h=1}^{t}h\lambda_h\sum_{j=0}^{h-1}\frac{(-1)^j}{c^{\alpha(j+2)}}\binom{h-1}{j}\right.$$
$$\left.\times\frac{p}{\{\alpha(j+2)-p\}}\frac{n}{n-r+1}\left\{\mu_{r:n}^{\alpha(j+2)-p}-\mu_{r-1:n}^{\alpha(j+2)-p}\right\}\right]. \quad \ldots(27)$$

We will now obtain the recurrence relations for product and ratio moments of *dgos* for general transmuted power function distributions in the following section.

Recurrence relations for product and ratio moments

In this section the recurrence relations for product and ratio moments of *dgos* for the general transmuted power function distributions will be given. These relations are obtained by considering the following relation between product moments of *dgos*, see for example Kotb *et al.* (2003) and Shahbaz *et al.* (2016).

$$\mu_{r,s:n,m,k}^{p,q}-\mu_{r,s-1:n,m,k}^{p,q}=-\frac{qC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{-\infty}^{x_1}x_1^p x_2^{q-1}f(x_1)\left[F(x_1)\right]^m$$
$$\times g_m^{r-1}\left[F(x_1)\right]\left[h_m(x_1)-h_m(x_2)\right]^{s-r-1}\left[F(x_2)\right]^{\gamma_s}dx_2dx_1 \quad \ldots(28)$$

where $\mu_{r,s:n,m,k}^{p,q} = E\left(X_{r:n,m,k}^{p} X_{s:n,m,k}^{q}\right)$. The recurrence relations for general transmuted power function distributions are obtained in the following subsections.

General transmuted power function distribution-I

In the following we will obtain the recurrence relations for product and ratio moments of *dgos* for *GTPF-I*. The recurrence relation for product moments of *dgos* from *GTPF-I* is given in the following Theorem.

Theorem 3: The product moments of general transmuted power function distribution-I are related as

$$\mu_{r,s:n,m,k}^{p,q} = \left(\frac{\alpha\gamma_s}{\alpha\gamma_s + q}\right)\Bigg[\mu_{r,s-1:n,m,k}^{p,q} + \sum_{h=1}^{t} \frac{\lambda_h q \gamma_{s(k-1)} C_{s-1}}{c^{\alpha(h+1)} \gamma_s C_{s-1(k-1)}} \left\{\frac{h}{q+\alpha(h+1)}\right.$$
$$\left.\times\left[\mu_{r,s:n,m,k-1}^{p,q+\alpha(h+1)} - \mu_{r,s-1:n,m,k-1}^{p,q+\alpha(h+1)}\right] - \frac{c^{\alpha}(h-1)}{q+\alpha h}\left[\mu_{r,s:n,m,k-1}^{p,q+\alpha h} - \mu_{r,s-1:n,m,k-1}^{p,q+\alpha h}\right]\right\}\Bigg]. \quad \ldots(29)$$

Proof: Using (11) in (28) we have

$$\mu_{r,s:n,m,k}^{p,q} - \mu_{r,s-1:n,m,k}^{p,q} = -\frac{q C_{s-1}}{\gamma_s (r-1)!(s-r-1)!} \int_0^c \int_0^{x_1} x_1^p x_2^{q-1} f(x_1)\left[F(x_1)\right]^m g_m^{r-1}\left[F(x_1)\right]$$
$$\times\left[h_m(x_1) - h_m(x_2)\right]^{s-r-1}\left[F(x_2)\right]^{\gamma_s - 1}$$
$$\times\left\{\frac{x}{\alpha} f_1(x_2) - \sum_{h=1}^{t}(h-1)\lambda_h \frac{x_2^{\alpha h}}{c^{\alpha h}} + \sum_{h=1}^{t} h\lambda_h \frac{x_2^{\alpha(h+1)}}{c^{\alpha(h+1)}}\right\} dx_2 dx_1$$

or

$$\mu_{r,s:n,m,k}^{p,q} - \mu_{r,s-1:n,m,k}^{p,q} = -\frac{q}{\alpha\gamma_s}\mu_{r,s:n,m,k}^{p,q} + \sum_{h=1}^{t} \frac{(h-1)\lambda_h q C_{s-1}}{\gamma_s c^{\alpha h}(r-1)!(s-r-1)!} \int_0^c \int_0^{x_1} x_1^p x_2^{q+\alpha h-1} f(x_1)$$
$$\times\left[F(x_1)\right]^m g_m^{r-1}\left[F(x_1)\right]\left[h_m(x_1) - h_m(x_2)\right]^{s-r-1}\left[F(x_2)\right]^{\gamma_s-1} dx_2 dx_1$$
$$-\sum_{h=1}^{t} \frac{h\lambda_h q C_{s-1}}{\gamma_s c^{\alpha h}(r-1)!(s-r-1)!} \int_0^c \int_0^{x_1} x_1^p x_2^{q+\alpha(h+1)-1} f(x_1)\left[F(x_1)\right]^m$$
$$\times g_m^{r-1}\left[F(x_1)\right]\left[h_m(x_1) - h_m(x_2)\right]^{s-r-1}\left[F(x_2)\right]^{\gamma_s-1} dx_2 dx_1$$

or

$$\mu_{r,s:n,m,k}^{p,q} - \mu_{r,s-1:n,m,k}^{p,q} = -\frac{q}{\alpha\gamma_s}\mu_{r,s:n,m,k}^{p,q} - \sum_{h=1}^{t} \frac{(h-1)\lambda_h q \gamma_{s(k-1)} C_{s-1}}{\gamma_s c^{\alpha h}(q+\alpha h) C_{s-1(k-1)}}\left\{\mu_{r,s:n,m,k-1}^{p,q+\alpha h} - \mu_{r,s-1:n,m,k-1}^{p,q+\alpha h}\right\}$$
$$+\sum_{h=1}^{t} \frac{h\lambda_h q \gamma_{s(k-1)} C_{s-1}}{\gamma_s c^{\alpha(h+1)}\left\{q+\alpha(h+1)\right\} C_{s-1(k-1)}}\left\{\mu_{r,s:n,m,k-1}^{p,q+\alpha(h+1)} - \mu_{r,s-1:n,m,k-1}^{p,q+\alpha(h+1)}\right\}.$$

Re-arranging the above equation we have (29) and hence the Theorem.

It is to be noted that the recurrence relations for product moments for *GTPF-I*, given in (29), reduces to the recurrence relations for product moments of *dgos* for power function distribution, given by Athar and Faizan (2011), for $\lambda_h = 0; h = 1, 2, \ldots, t$.

Some useful corollaries which are immediate from the above Theorem are given below.

Corollary 7: Using "$-q$" in place of q in (28) the following relation for ratio moments of *dgos* for *GTPF-I* is obtained

$$\mu_{r,s:n,m,k}^{p,-q} = \left(\frac{\alpha\gamma_s}{\alpha\gamma_s - q}\right)\left[\mu_{r,s-1:n,m,k}^{p,-q} - \sum_{h=1}^{t}\frac{\lambda_h q\gamma_{s(k-1)}C_{s-1}}{c^{\alpha(h+1)}\gamma_s C_{s-1(k-1)}}\left\{\frac{h}{\alpha(h+1)-q}\right.\right.$$
$$\left.\left.\times\left[\mu_{r,s:n,m,k-1}^{p,\alpha(h+1)-q} - \mu_{r,s-1:n,m,k-1}^{p,\alpha(h+1)-q}\right] - \frac{c^{\alpha}(h-1)}{\alpha h - q}\left[\mu_{r,s:n,m,k-1}^{p,\alpha h-q} - \mu_{r,s-1:n,m,k-1}^{p,\alpha h-q}\right]\right\}\right]. \quad \text{....(30)}$$

Corollary 8: Using $m = -1$ in (29) and (30) the following relations for product and ratio moments of lower record values for *GTPF-I* are obtained

$$\mu_{K(r,s)}^{p,q} = \left(\frac{\alpha k}{\alpha k + q}\right)\left[\mu_{K(r,s-1)}^{p,q} + \sum_{h=1}^{t}\frac{\lambda_h q}{c^{\alpha(h+1)}}\left(\frac{k}{k-1}\right)^{s-1}\left\{\frac{h}{q+\alpha(h+1)}\right.\right.$$
$$\left.\left.\times\left[\mu_{K-1(r,s)}^{p,q+\alpha(h+1)} - \mu_{K-1(r,s-1)}^{p,q+\alpha(h+1)}\right] - \frac{c^{\alpha}(h-1)}{q+\alpha h}\left[\mu_{K-1(r,s)}^{p,q+\alpha h} - \mu_{K-1(r,s-1)}^{p,q+\alpha h}\right]\right\}\right]. \quad \text{...(31)}$$

$$\mu_{K(r,s)}^{p,-q} = \left(\frac{\alpha k}{\alpha k + q}\right)\left[\mu_{K(r,s-1)}^{p,-q} - \sum_{h=1}^{t}\frac{\lambda_h q}{c^{\alpha(h+1)}}\left(\frac{k}{k-1}\right)^{s-1}\left\{\frac{h}{\alpha(h+1)-q}\right.\right.$$
$$\left.\left.\times\left[\mu_{K-1(r,s)}^{p,\alpha(h+1)-q} - \mu_{K-1(r,s-1)}^{p,\alpha(h+1)-q}\right] - \frac{c^{\alpha}(h-1)}{\alpha h - q}\left[\mu_{K-1(r,s)}^{p,\alpha h-q} - \mu_{K-1(r,s-1)}^{p,\alpha h-q}\right]\right\}\right]. \quad \text{...(32)}$$

Corollary 9: Using $m = 0$ and $k = 1$ in (29) and (30) the following relations for product and ratio moments of reversed order statistics for *GTPF-I* are obtained

$$\mu_{r,s:n}^{p,q} = \left[\frac{\alpha(n-s+1)}{\alpha(n-s+1)+q}\right]\left[\mu_{r,s-1:n}^{p,q} + \sum_{h=1}^{t}\frac{\lambda_h q}{c^{\alpha(h+1)}}\frac{n}{n-s+1}\left\{\frac{h}{q+\alpha(h+1)}\right.\right.$$
$$\left.\left.\times\left[\mu_{r,s:n}^{p,q+\alpha(h+1)} - \mu_{r,s-1:n}^{p,q+\alpha(h+1)}\right] - \frac{c^{\alpha}(h-1)}{q+\alpha h}\left[\mu_{r,s:n}^{p,q+\alpha h} - \mu_{r,s-1:n}^{p,q+\alpha h}\right]\right\}\right]. \quad \text{...(33)}$$

$$\mu_{r,s:n}^{p,-q} = \left[\frac{\alpha(n-s+1)}{\alpha(n-s+1)-q}\right]\left[\mu_{r,s-1:n}^{p,-q} - \sum_{h=1}^{t}\frac{\lambda_h q}{c^{\alpha(h+1)}}\frac{n}{n-s+1}\left\{\frac{h}{\alpha(h+1)-q}\right.\right.$$
$$\left.\left.\times\left[\mu_{r,s:n}^{p,\alpha(h+1)-q} - \mu_{r,s-1:n}^{p,\alpha(h+1)-q}\right] - \frac{c^{\alpha}(h-1)}{\alpha h - q}\left[\mu_{r,s:n}^{p,\alpha h-q} - \mu_{r,s-1:n}^{p,\alpha h-q}\right]\right\}\right]. \quad \text{...(34)}$$

We will now give recurrence relations for product and ratio moments of *dgos* for *GTPF-II* in the following.

General transmuted power function distribution–II

In the following Theorem the recurrence relations for product and ratio moments of *dgos* for *GTPF-II* are given.

Theorem 4: The product moments of general transmuted power function distribution-II are related as

$$\mu_{r,s:n,m,k}^{p,q} = \left(\frac{\alpha\gamma_s}{\alpha\gamma_s + q}\right)\left[\mu_{r,s-1:n,m,k}^{p,q} + \sum_{h=1}^{t}h\lambda_h\sum_{j=0}^{h-1}(-1)^j\binom{h-1}{j}\right.$$
$$\left.\times\frac{q\gamma_{s(k-1)}C_{s-1}}{\gamma_s c^{\alpha(j+2)}\{q+\alpha(j+2)\}C_{s-1(k-1)}}\left\{\mu_{r,s:n,m,k-1}^{p,q+\alpha(j+2)} - \mu_{r,s-1:n,m,k-1}^{p,q+\alpha(j+2)}\right\}\right]. \quad \text{...(35)}$$

Proof: Using (14) in (28) we have

$$\mu_{r,s:n,m,k}^{p,q} - \mu_{r,s-1:n,m,k}^{p,q} = -\frac{qC_{s-1}}{\gamma_s (r-1)!(s-r-1)!}\int_0^c\int_0^{x_1} x_1^p x_2^{q-1} f(x_1)\left[F(x_1)\right]^m g_m^{r-1}\left[F(x_1)\right]$$
$$\times\left[h_m(x_1)-h_m(x_2)\right]^{s-r-1}\left[F(x_2)\right]^{\gamma_s-1}$$
$$\times\left\{\frac{x}{\alpha}f_2(x_2)+\sum_{h=1}^{t}h\lambda_h\sum_{j=0}^{h-1}(-1)^j\binom{h-1}{j}\frac{x_2^{\alpha(j+2)}}{c^{\alpha(j+2)}}\right\}dx_2dx_1$$
$$= -\frac{q}{\alpha\gamma_s}\mu_{r,s:n,m,k}^{p,q} + \sum_{h=1}^{t}h\lambda_h\sum_{j=0}^{h-1}(-1)^j\binom{h-1}{j}\frac{qC_{s-1}}{\gamma_s c^{\alpha(j+2)}(r-1)!(s-r-1)!}$$
$$\times\int_0^c\int_0^{x_1} x_1^p x_2^{q+\alpha(j+2)-1} f(x_1)\left[F(x_1)\right]^m g_m^{r-1}\left[F(x_1)\right]$$
$$\times\left[h_m(x_1)-h_m(x_2)\right]^{s-r-1}\left[F(x_2)\right]^{\gamma_s-1}dx_2dx_1$$

or

$$\mu_{r,s:n,m,k}^{p,q} - \mu_{r,s-1:n,m,k}^{p,q} = -\frac{q}{\alpha\gamma_s}\mu_{r,s:n,m,k}^{p,q} + \sum_{h=1}^{t}h\lambda_h\sum_{j=0}^{h-1}(-1)^j\binom{h-1}{j}$$
$$\times\frac{q\gamma_{s(k-1)}C_{s-1}}{\gamma_s c^{\alpha(j+2)}\left\{q+\alpha(j+2)\right\}C_{s-1(k-1)}}\left\{\mu_{r,s:n,m,k-1}^{p,q+\alpha(j+2)} - \mu_{r,s-1:n,m,k-1}^{p,q+\alpha(j+2)}\right\}.$$

Re-arranging, we have (35) and hence the theorem.

The relation (35) again reduces to the recurrence relation for product moments of *dgos* for power function distribution, given by Athar and Faizan (2011), for $\lambda_h = 0; h = 1,2,\ldots,t$. The corollaries which follow from Theorem 4 are given below.

Corollary 10: Using "–*q*" in place of *q* in (35) the following relation for ratio moments of *dgos* for *GTPF-II* is obtained

$$\mu_{r,s:n,m,k}^{p,-q} = \left(\frac{\alpha\gamma_s}{\alpha\gamma_s - q}\right)\left[\mu_{r,s-1:n,m,k}^{p,-q} - \sum_{h=1}^{t}h\lambda_h\sum_{j=0}^{h-1}(-1)^j\binom{h-1}{j}\right.$$
$$\left.\times\frac{q\gamma_{s(k-1)}C_{s-1}}{\gamma_s c^{\alpha(j+2)}\left\{\alpha(j+2)-q\right\}C_{s-1(k-1)}}\left\{\mu_{r,s:n,m,k-1}^{p,\alpha(j+2)-q} - \mu_{r,s-1:n,m,k-1}^{p,\alpha(j+2)-q}\right\}\right]. \quad (36)$$

Corollary 11: Using *m* = –1 in (35) and (36) the following relations for product and ratio moments of lower record values for *GTPF-II* are obtained

$$\mu_{K(r,s)}^{p,q} = \left(\frac{\alpha k}{\alpha k + q}\right)\left[\mu_{K(r,s-1)}^{p,q} + \sum_{h=1}^{t}h\lambda_h\sum_{j=0}^{h-1}(-1)^j\binom{h-1}{j}\right.$$
$$\left.\times\frac{q}{c^{\alpha(j+2)}\left\{q+\alpha(j+2)\right\}}\left(\frac{k}{k-1}\right)^{s-1}\left\{\mu_{K-1(r,s)}^{p,q+\alpha(j+2)} - \mu_{K-1(r,s-1)}^{p,q+\alpha(j+2)}\right\}\right]. \quad \ldots(37)$$

$$\mu_{K(r,s)}^{p,-q} = \left(\frac{\alpha k}{\alpha k - q}\right)\left[\mu_{K(r,s-1)}^{p,-q} - \sum_{h=1}^{t}h\lambda_h\sum_{j=0}^{h-1}(-1)^j\binom{h-1}{j}\right.$$
$$\left.\times\frac{q}{c^{\alpha(j+2)}\left\{\alpha(j+2)-q\right\}}\left(\frac{k}{k-1}\right)^{s-1}\left\{\mu_{K-1(r,s)}^{p,\alpha(j+2)-q} - \mu_{K-1(r,s-1)}^{p,\alpha(j+2)-q}\right\}\right]. \quad \ldots(38)$$

Corollary 12: Using $m = 0$ and $k = 1$ in (35) and (36) the following relations for product and ratio moments of reversed order statistics for *GTPF-II* are obtained

$$\mu_{r,s:n}^{p,q} = \left[\frac{\alpha(n-s+1)}{\alpha(n-s+1)+q}\right]\left[\mu_{r,s-1:n}^{p,q} + \sum_{h=1}^{t} h\lambda_h \sum_{j=0}^{h-1}(-1)^j \binom{h-1}{j} \times \frac{q}{c^{\alpha(j+2)}\{q+\alpha(j+2)\}}\frac{n}{n-s+1}\left\{\mu_{r,s:n}^{p,q+\alpha(j+2)} - \mu_{r,s-1:n}^{p,q+\alpha(j+2)}\right\}\right]. \quad \ldots(39)$$

$$\mu_{r,s:n}^{p,-q} = \left[\frac{\alpha(n-s+1)}{\alpha(n-s+1)-q}\right]\left[\mu_{r,s-1:n}^{p,-q} - \sum_{h=1}^{t} h\lambda_h \sum_{j=0}^{h-1}(-1)^j \binom{h-1}{j} \times \frac{q}{c^{\alpha(j+2)}\{\alpha(j+2)-q\}}\frac{n}{n-s+1}\left\{\mu_{r,s:n}^{p,\alpha(j+2)-q} - \mu_{r,s-1:n}^{p,\alpha(j+2)-q}\right\}\right]. \quad \ldots(40)$$

We will now give some characterizations of general transmuted power function distributions based upon single and product moment of *dgos*.

Characterizations

In this section some characterization results will be presented for general transmuted power function distributions. These characterization results are based upon single and product moments of *dgos*.

General transmuted power function-I

The characterization results for *GTPF-I* are given in the following. These characterizations are given in the following Theorems.

Theorem 5: A necessary and sufficient condition for a random variable X to have density and distribution functions (9) and (10), respectively, is that the moments of its *dgos* are related as

$$\mu_{r:n,m,k}^{p} = \left(\frac{\alpha\gamma_r}{\alpha\gamma_r + p}\right)\left[\mu_{r-1:n,m,k}^{p} + \sum_{h=1}^{t} \frac{\lambda_h p \gamma_{r(k-1)} C_{r-1}}{c^{\alpha(h+1)}\gamma_r C_{r-1(k-1)}}\left\{\frac{h}{p+\alpha(h+1)} \times\left[\mu_{r:n,m,k-1}^{p+\alpha(h+1)} - \mu_{r-1:n,m,k-1}^{p+\alpha(h+1)}\right] - \frac{c^{\alpha}(h-1)}{p+\alpha h}\left[\mu_{r:n,m,k-1}^{p+\alpha h} - \mu_{r-1:n,m,k-1}^{p+\alpha h}\right]\right\}\right].$$

Proof: The necessary condition immediately follows from Theorem 1. For a sufficient condition consider (15) as

$$\mu_{r:n,m,k}^{p} - \mu_{r-1:n,m,k}^{p} = -\frac{pC_{r-1}}{\gamma_r (r-1)!}\int_{-\infty}^{\infty} x^{p-1}\left[F(x)\right]^{\gamma_r} g_m^{r-1}\left[F(x)\right]dx .$$

Using the above equation with (11) we have

$$-\frac{pC_{r-1}}{\gamma_r (r-1)!}\int_{-\infty}^{\infty} x^{p-1}\left[F(x)\right]^{\gamma_r} g_m^{r-1}\left[F(x)\right]dx = -\frac{pC_{r-1}}{\gamma_r (r-1)!}\int_{-\infty}^{\infty} x^{p-1} \times\left\{\frac{x}{\alpha}f_1(x) - \sum_{h=1}^{t}(h-1)\lambda_h \frac{x^{\alpha h}}{c^{\alpha h}} + \sum_{h=1}^{t} h\lambda_h \frac{x^{\alpha(h+1)}}{c^{\alpha(h+1)}}\right\}\left[F(x)\right]^{\gamma_r - 1} g_m^{r-1}\left[F(x)\right]dx$$

or

$$-\frac{pC_{r-1}}{\gamma_r(r-1)!}\int_{-\infty}^{\infty} x^{p-1}\left[F(x)\right]^{\gamma_r-1} g_m^{r-1}\left[F(x)\right]$$
$$\times\left[F(x)-\left\{\frac{x}{\alpha}f(x)-\sum_{h=1}^{t}(h-1)\lambda_h\frac{x^{\alpha h}}{c^{\alpha h}}+\sum_{h=1}^{t}h\lambda_h\frac{x^{\alpha(h+1)}}{c^{\alpha(h+1)}}\right\}\right]=0.$$

Applying the *Müntz–Száz* theorem—see Hwang and Lin (1984)—to the above equation we have

$$F(x)=\frac{x}{\alpha}f(x)-\sum_{h=1}^{t}(h-1)\lambda_h\frac{x^{\alpha h}}{c^{\alpha h}}+\sum_{h=1}^{t}h\lambda_h\frac{x^{\alpha(h+1)}}{c^{\alpha(h+1)}}$$

and this relation holds between the density and distribution function of *GTPF-I* distribution and hence the theorem.

Theorem 6: A necessary and sufficient condition for a random variable X to have density and distribution functions (9) and (10), respectively, is that the joint moments of its *dgos* are related as

$$\mu_{r,s:n,m,k}^{p,q}=\left(\frac{\alpha\gamma_s}{\alpha\gamma_s+q}\right)\left[\mu_{r,s-1:n,m,k}^{p,q}+\sum_{h=1}^{t}\frac{\lambda_h q\gamma_{s(k-1)}C_{s-1}}{c^{\alpha(h+1)}\gamma_s C_{s-1(k-1)}}\left\{\frac{h}{q+\alpha(h+1)}\right.\right.$$
$$\left.\left.\times\left[\mu_{r,s:n,m,k-1}^{p,q+\alpha(h+1)}-\mu_{r,s-1:n,m,k-1}^{p,q+\alpha(h+1)}\right]-\frac{c^{\alpha}(h-1)}{q+\alpha h}\left[\mu_{r,s:n,m,k-1}^{p,q+\alpha h}-\mu_{r,s-1:n,m,k-1}^{p,q+\alpha h}\right]\right\}\right].$$

Proof: The necessary condition immediately follows from Theorem 3. For a sufficient condition consider (28) as

$$\mu_{r,s:n,m,k}^{p,q}-\mu_{r,s-1:n,m,k}^{p,q}=-\frac{qC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{-\infty}^{x_1}x_1^p x_2^{q-1}f(x_1)\left[F(x_1)\right]^m$$
$$\times g_m^{r-1}\left[F(x_1)\right]\left[h_m(x_1)-h_m(x_2)\right]^{s-r-1}\left[F(x_2)\right]^{\gamma_s}dx_2dx_1$$

Using the above with (11) we have

$$-\frac{qC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_0^{\infty}\int_0^{x_1}x_1^p x_2^{q-1}f(x_1)\left[F(x_1)\right]g_m^{r-1}\left[F(x_1)\right]^m$$
$$\times\left[h_m(x_1)-h_m(x_2)\right]^{s-r-1}\left[F(x_2)\right]^{\gamma_s}dx_2dx_1=-\frac{qC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}$$
$$\times\int_0^{\infty}\int_0^{x_1}x_1^p x_2^{q-1}f(x_1)\left[F(x_1)\right]g_m^{r-1}\left[F(x_1)\right]^m\left[h_m(x_1)-h_m(x_2)\right]^{s-r-1}$$
$$\times\left[F(x_2)\right]^{\gamma_s-1}\left\{\frac{x}{\alpha}f(x_2)-\sum_{h=1}^{t}(h-1)\lambda_h\frac{x_2^{\alpha h}}{c^{\alpha h}}+\sum_{h=1}^{t}h\lambda_h\frac{x_2^{\alpha(h+1)}}{c^{\alpha(h+1)}}\right\}dx_2dx_1$$

or

$$-\frac{qC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_0^{\infty}\int_0^{x_1}x_1^p x_2^{q-1}f(x_1)\left[F(x_1)\right]g_m^{r-1}\left[F(x_1)\right]^m\left[h_m(x_1)-h_m(x_2)\right]^{s-r-1}$$
$$\times\left[F(x_2)\right]^{\gamma_s-1}\left[F(x)-\left\{\frac{x}{\alpha}f(x_2)-\sum_{h=1}^{t}(h-1)\lambda_h\frac{x_2^{\alpha h}}{c^{\alpha h}}+\sum_{h=1}^{t}h\lambda_h\frac{x_2^{\alpha(h+1)}}{c^{\alpha(h+1)}}\right\}\right]dx_2dx_1=0.$$

Applying the *Müntz–Száz* theorem—see Hwang and Lin (1984)—to the above equation we have

$$F(x_2)=\frac{x_2}{\alpha}f(x_2)-\sum_{h=1}^{t}(h-1)\lambda_h\frac{x_2^{\alpha h}}{c^{\alpha h}}+\sum_{h=1}^{t}h\lambda_h\frac{x_2^{\alpha(h+1)}}{c^{\alpha(h+1)}}$$

and this relation holds between density and distribution function of *GTPF-I* distribution and hence the theorem.

General transmuted power function-II

The characterization results for *GTPF-II* are given in following Theorems

Theorem 7: A necessary and sufficient condition for a random variable X to have density and distribution functions (12) and (13), respectively, is that the moments of its *dgos* are related as

$$\mu_{r:n,m,k}^{p} = \left(\frac{\alpha\gamma_r}{\alpha\gamma_r + p}\right)\left[\mu_{r-1:n,m,k}^{p} + \sum_{h=1}^{t} h\lambda_h \sum_{j=0}^{h-1} \frac{(-1)^j}{c^{\alpha(j+2)}}\binom{h-1}{j} \times \frac{p\gamma_{r(k-1)}C_{r-1}}{\{p+\alpha(j+2)\}\gamma_r C_{r-1(k-1)}}\left\{\mu_{r:n,m,k-1}^{p+\alpha(j+2)} - \mu_{r-1:n,m,k-1}^{p+\alpha(j+2)}\right\}\right]$$

Proof: The necessary condition immediately follow from Theorem 2. For a sufficient condition consider (15) as

$$\mu_{r:n,m,k}^{p} - \mu_{r-1:n,m,k}^{p} = -\frac{pC_{r-1}}{\gamma_r (r-1)!}\int_{-\infty}^{\infty} x^{p-1}\left[F(x)\right]^{\gamma_r} g_m^{r-1}\left[F(x)\right]dx .$$

Using the above equation with (14) we have

$$-\frac{pC_{r-1}}{\gamma_r (r-1)!}\int_{-\infty}^{\infty} x^{p-1}\left[F(x)\right]^{\gamma_r} g_m^{r-1}\left[F(x)\right]dx = -\frac{pC_{r-1}}{\gamma_r (r-1)!}\int_{-\infty}^{\infty} x^{p-1}\left[F(x)\right]^{\gamma_r -1} \times\left\{\frac{x}{\alpha}f_2(x) + \sum_{h=1}^{t} h\lambda_h \sum_{j=0}^{h-1}(-1)^j\binom{h-1}{j}\frac{x^{\alpha(j+2)}}{c^{\alpha(j+2)}}\right\} g_m^{r-1}\left[F(x)\right]dx$$

or

$$-\frac{pC_{r-1}}{\gamma_r (r-1)!}\int_{-\infty}^{\infty} x^{p-1}\left[F(x)\right]^{\gamma_r -1} g_m^{r-1}\left[F(x)\right] \times\left[F(x) - \left\{\frac{x}{\alpha}f(x) + \sum_{h=1}^{t} h\lambda_h \sum_{j=0}^{h-1}(-1)^j\binom{h-1}{j}\frac{x^{\alpha(j+2)}}{c^{\alpha(j+2)}}\right\}\right]dx = 0.$$

Applying the *Müntz–Száz* theorem—see Hwang and Lin (1984)—to the above equation we have

$$F(x) = \frac{x}{\alpha}f(x) + \sum_{h=1}^{t} h\lambda_h \sum_{j=0}^{h-1}(-1)^j\binom{h-1}{j}\frac{x^{\alpha(j+2)}}{c^{\alpha(j+2)}}$$

and this relation holds between density and distribution function of *GTPF-II* distribution and hence the theorem.

Theorem 8: A necessary and sufficient condition for a random variable X to have density and distribution functions (12) and (13), respectively, is that the joint moments of its *dgos* are related as

$$\mu_{r,s:n,m,k}^{p,q} = \left(\frac{\alpha\gamma_s}{\alpha\gamma_s + q}\right)\left[\mu_{r,s-1:n,m,k}^{p,q} + \sum_{h=1}^{t} h\lambda_h \sum_{j=0}^{h-1}(-1)^j\binom{h-1}{j} \times \frac{q\gamma_{s(k-1)}C_{s-1}}{\gamma_s c^{\alpha(j+2)}\{q+\alpha(j+2)\}C_{s-1(k-1)}}\left\{\mu_{r,s:n,m,k-1}^{p,q+\alpha(j+2)} - \mu_{r,s-1:n,m,k-1}^{p,q+\alpha(j+2)}\right\}\right].$$

Proof: The necessary condition immediately follow from Theorem 3. For a sufficient condition consider (28) as

$$\mu_{r,s:n,m,k}^{p,q} - \mu_{r,s-1:n,m,k}^{p,q} = -\frac{qC_{s-1}}{\gamma_s (r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{-\infty}^{x_1} x_1^{p} x_2^{q-1} f(x_1)\left[F(x_1)\right]^{m} \times g_m^{r-1}\left[F(x_1)\right]\left[h_m(x_1) - h_m(x_2)\right]^{s-r-1}\left[F(x_2)\right]^{\gamma_s} dx_2 dx_1 .$$

Using the above with (14) we have

$$-\frac{qC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{-\infty}^{x_1} x_1^p x_2^{q-1} f(x_1)\left[F(x_1)\right]^m g_m^{r-1}\left[F(x_1)\right]$$
$$\times\left[h_m(x_1)-h_m(x_2)\right]^{s-r-1}\left[F(x_2)\right]^{\gamma_s} dx_2 dx_1 = -\frac{qC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}$$
$$\times\int_{-\infty}^{\infty}\int_{-\infty}^{x_1} x_1^p x_2^{q-1} f(x_1)\left[F(x_1)\right]^m g_m^{r-1}\left[F(x_1)\right]\left[h_m(x_1)-h_m(x_2)\right]^{s-r-1}$$
$$\times\left[F(x_2)\right]^{\gamma_s-1}\left\{\frac{x_2}{\alpha}f(x_2)+\sum_{h=1}^{t}h\lambda_h\sum_{j=0}^{h-1}(-1)^j\binom{h-1}{j}\frac{x_2^{\alpha(j+2)}}{c^{\alpha(j+2)}}\right\}dx_2 dx_1$$

or

$$-\frac{qC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{-\infty}^{x_1} x_1^p x_2^{q-1} f(x_1)\left[F(x_1)\right]^m g_m^{r-1}\left[F(x_1)\right]\left[h_m(x_1)-h_m(x_2)\right]^{s-r-1}$$
$$\times\left[F(x_2)\right]^{\gamma_s-1}\left[F(x_2)-\left\{\frac{x_2}{\alpha}f(x_2)+\sum_{h=1}^{t}h\lambda_h\sum_{j=0}^{h-1}(-1)^j\binom{h-1}{j}\frac{x_2^{\alpha(j+2)}}{c^{\alpha(j+2)}}\right\}\right]dx_2 dx_1 .$$

Applying the *Müntz–Száz* theorem—see Hwang and Lin (1984)—to above equation we have

$$F(x_2)=\frac{x_2}{\alpha}f(x_2)+\sum_{h=1}^{t}h\lambda_h\sum_{j=0}^{h-1}(-1)^j\binom{h-1}{j}\frac{x_2^{\alpha(j+2)}}{c^{\alpha(j+2)}}$$

and this relation holds between density and distribution function of *GTPF-II* distribution and hence the theorem.

CONCLUSION

In this paper some recurrence relations for single, inverse, product and ratio moments of *dgos* for the general transmuted power function distribution are obtained. These relations are helpful for computing higher order moments from the lower order moments. These relations also enable us to compute moments of special cases of *dgos* from the general transmuted power function distribution. These relations are also helpful to compute moments of *dgos* and its special cases for any order transmuted distribution for power function baseline distribution.

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