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FIRST ORDER NECESSARY AND SUFFICIENT CONDITIONS FOR MATHEMATICAL PROGRAMMING WITH *n*-SET FUNCTIONS

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ABSTRACT

In this paper we define the convexity of a collection of sets, convexity of an *n*-set function, and differentiability of an *n*-set function. We also give an example of a differentiable *n*-set function. Some basic properties of differentiable convex *n*-set functions are given in this paper. Some of these fundamental properties will be used to derive an optimization theorem involving convex *n*-set functions. Finally, we consider n-set functions and obtain valuable results that can be used extensively in convex programming. We conclude this paper with some results for vector valued convex *n*-set functions.

1. INTRODUCTION

Throughout this dissertation let (χ, A, μ) be a finite atomless measure space, that is, for each $\Omega \in A$ with $\mu\Omega > 0$, there exists $\Omega' \subset \Omega, \Omega \in A$ with $0 < \mu\Omega' < \mu\Omega$, and the space $L_1(\chi, A, \mu)$ is separable, that is, it has a countable dense subset. We will be concerned with *n*-set functions on A^n which is defined by

$$A^n = \{(S_1, \ldots S_n) : S_i \in A,\ i = 1, \ldots, n\}$$

Note that A^n is a pseudometric space under the pseudometric d defined by

$$d[(R_1, \ldots, R_n), (S_1, \ldots, S_n)] = \left\{\sum_{i=1}^{n} \mu^2(R_i \Delta S_i)\right\}^{\frac{1}{2}}, \quad R_i, S_i \in A,$$

where $R_i \Delta S_i = (R_i \setminus S_i) \cup (S_i \setminus R_i),\ i = 1, \ldots, n$. According to the assumption that (χ, A, μ) is a finite measure space, we can identify any set $S \in A$ with its characteristic function $\chi_S \in L_1(\chi, A, \mu)$. Note that $d[(R_1, \ldots, R_n), (S_1, \ldots, S_n)] =$ $\left\{\sum_{i=1}^{n} \left\|\chi_{R_i} - \chi_{S_i}\right\|_{L_1}^2\right\}^{\frac{1}{2}}$. For $f \in L_1(\chi, A, \mu)$ and $S \in A$, the integral $\int_S f\, d\mu$ will be denoted by $\langle f, \chi_S \rangle$.

2. RESULTS AND DISCUSSION

Morris[1] showed that if (χ, A, μ) is finite and atomless and $L_1(\chi, A, \mu)$ is separable, then for any $R, S \in A$ and $\lambda \in [0,1]$, there exist L_∞ – sequences $\{\chi_{R^l(\lambda)}\}\{\chi_{S^l(\lambda)}\}$ such that

$$\chi_{R^l(\lambda)} \xrightarrow{w^*} \lambda\chi_{R\setminus S}, \qquad \chi_{S^l(\lambda)} \xrightarrow{w^*} (1-\lambda)\chi_{S\setminus R},$$

and

$$\chi_{R^l(\lambda)\cup S^l(\lambda)\cup(R\cap S)} \xrightarrow{w^*} \lambda\chi_R + (1-\lambda)\chi_S,$$

where $\xrightarrow{w^*}$ stands for the *weak** - convergence. We shall call the sequence $\{V^l(\lambda)\}$ in A, where $V^l(\lambda) = R^l(\lambda) \cup S^l(\lambda) \cup (R \cap S)$, a Morris sequence associated with $\langle\lambda, R, S\rangle$. The definition of a convex subfamily S of A was first given by the authors in[2]. The following definition of a convex subfamily S^n of A^n, the n- fold product of S, is the extension of the one given in[2]. This definition will be used in the later part of this paper to characterize a convex n-set function by a convex subfamily. The author would like to note that the notion of convex subfamily is also defined with slight variation in[3].

Definition 2.1

A subfamily S^n of A^n is said to be convex if, for any $(R_1,\ldots,R_n)$, $(S_1,\ldots,S_n) \in S^n$, $\lambda \in [0,1]$ and every Morris sequence $\{V_i^l(\lambda)\} \subset A$ associated with $\langle\lambda_i, R_i, S_i\rangle$., for each $i = 1,\ldots,n$, there exists a subsequence $\{V_i^{l_k}(\lambda)\}$ of $\{V_i^l(\lambda)\}$ in S.

The following is an example of a convex subfamily.

Example 2.1

The subfamily $S^n \subset A^n$ defined by

$$S^n = \{(S_1,\ldots,S_n) \in A^n : \sum_{i=1}^{n} \langle v_i, \chi_{S_i}\rangle < \alpha,\ \alpha \in R,\ v_i \in L_1(\chi, A, \mu),\ i = 1,\ldots,n\}$$

is convex.

Since a measure space (χ, A, μ) need not have a linear structure, the notion convexity of a set function is different from the usual one. Initially the notion of a convex set function was given in[1] and it was extended to n-set functions by Corley[4] to the following form. Lin[3] defined the notion of convex n-set function in a different setting.

Definition 2.2

An n-set function $F : S^n \to R$ is said to be a convex subfamily S^n of A^n *if for each $\lambda \in [0,1]$ and $(R_1,\ldots,R_n),(S_1,\ldots,S_n) \in S^n$,*

$\limsup_{l\to\infty} F(V_1^l(\lambda),\ldots,V_n^l(\lambda)) \le \lambda F(R_1,\ldots,R_n) + (1-\lambda)F(S_1,\ldots,S_n)$ *for any Morris sequence $\{V_i^l(\lambda)\} \subset S$ associated with $\langle\lambda, R_i, S_i\rangle$. for each $i = 1,\ldots,n$. F is said to be a concave n-set function if –F is a convex n-set function.*

We shall next define the notion of differentiability of a function of a single set which was originally introduced by Morris[1]. We also shall extend it to the differentiability of an n-set function after the i th partial derivatives of an n-set function is defined.

Definition 2.3

A set function $H : A \to R$ *is differentiable at* $S \in A$ *if there exists an* $h_s \in L\left(\chi, A, \mu\right)$, *the derivative of H at S, such that*

$$H(R) = H(S) + \left\langle h_s, \chi_s - \chi_R \right\rangle + W_H(R,S),$$

where $W_H(R,S) \in o\left[d\left(R,S\right)\right]$, *that is,* $\lim_{d(R,S)\to 0} \dfrac{W_H(R,S)}{d(R,S)} = 0$.

H is said to be differentiable on A if H is differentiable at all $S \in A$.

The following definition of the ith partial derivative of an n-set function is due to Corley[4].

Definition 2.4

Let $F : A^n \to R$ *and* $(S_1,\ldots,S_n) \in A^n$. *Then S is said to have an i th partial derivative at* $\left(S_1,\ldots,S_n\right)$ *if the set function*

$$H(R_i) = F(S_i,\ldots,S_{i-1}, R_i, S_{i+1},\ldots S_n)$$

has a derivative h_{s_i} *at* S_i *and it is denoted by* f_*^i.

As we stated earlier in this paper, the finiteness of (χ, A, μ) allows A^n to be made into a pseudometric space. This enables us to define the differentiability of an n-set function which is also due to Corley[4]. This is a direct extension of the differentiability of a set function which was given earlier in this paper.

Definition 2.5

An n-set function $F : A^n \to R$ *is differentiable at* $(S_1,\ldots,S_n) \in A^n$ *if all the partial derivatives* f_*^i, $i = 1,\ldots,n$, *exist and satisfy*

$$F(R_1,\ldots,R_n) = F(S_1,\ldots,S_n) + \sum_{i=1}^{n}\left\langle f_*^i, \chi_{R_i} - \chi_{S_i}\right\rangle + W_F\left[(R_1,\ldots,R_n),(S_1,\ldots,S_n)\right],$$

where f_*^i *is the i th partial derivative of F at* $(S_1,\ldots,S_n)$ *and*

$W_F\left[(R_1,\ldots,R_n),(S_1,\ldots,S_n)\right] \in o\left[d\left[(R_1,\ldots,R_n),(S_1,\ldots,S_n)\right]\ \right]$.

F is said to be differentiable on A^n *if F is differentiable at all* $(S_1,\ldots,S_n) \in A^n$.

Example 2.2

An example of a differentiable convex n-set function is

$$F(R_1,\ldots,R_n) = h\left(\int_{R_1} v_1 d\mu,\ldots,\int_{R_n} v_n d\mu\right),$$

where $h : R^n \to R$ *is convex, differentiable, and* $v_i \in L_1(\chi, A, \mu)$ *for each* $i = 1,\ldots,n$.

For any $(R_1,\ldots,R_n),(S_1,\ldots,S_n) \in A^n$ and $\lambda \in \left[0,1\right]$, there exists a Morris sequence $\left\{V_i^l(\lambda)\right\}$ in A such that $\chi_{V_i^l(\lambda)} \xrightarrow{w^*} \lambda\chi_{R_i} + (1-\lambda)\chi_{S_i}$, for each $i = 1,\ldots,n$. Therefore

$$\begin{aligned}\limsup_{l\to\infty} F(V_1^l(\lambda),\ldots,V_n^l(\lambda)) &= \limsup_{l\to\infty} h\left(\int_{V_1^l(\lambda)} v_1 d\mu,\ldots,\int_{V_n^l(\lambda)} v_n d\mu\right) \\ &= h\left(\lambda\left(\int_{R_1} v_1 d\mu,\ldots,\int_{R_n} v_n d\mu\right) + \left(\int_{S_1} v_1 d\mu,\ldots,\int_{S_n} v_n d\mu\right)\right)\end{aligned}$$

$$\leq \lambda h\left(\int_{R_1} v_1 d\mu,\ldots,\int_{R_n} v_n d\mu\right)+(1-\lambda)h\left(\int_{S_1} v_1 d\mu,\ldots,\int_{S_n} v_n d\mu\right)$$

(since h is convex)

$$= \lambda F(R_1,\ldots,R_n)+(1-\lambda)F(S_1,\ldots,S_n).$$

This proves that F is a convex n-set function.

The i th partial derivative of F at $(S_1,\ldots,S_n)$ is given by

$$f_*^i = h^{(i)}\left(\int_{S_1} v_1 d\mu,\ldots,\int_{S_n} v_n d\mu\right) v_i,$$

where $h^{(i)}$ is the i th partial derivative of h.

Now we state the following lemma which will be used later in our study.

Lemma 2.1

(Theorem 3[3]) Let $F : A^n \to R$ be an n-set function, differentiable at $(S_1,\ldots,S_n) \in A^n$. For any given $(R_1,\ldots,R_n) \in A^n$ and $\lambda \in [0,1]$, let $\{V_i^l(\lambda)\} \subset A$ be a Morris sequence associated with $\langle \lambda, R_i, S_i \rangle$ for each $i = 1,\ldots,n$. Then

$$\limsup_{l\to\infty} W_F\left[\left(V_1^l(\lambda),\ldots,V_n^l(\lambda)\right),(S_1,\ldots,S_n)\right] = o(\lambda),$$

where $\lim_{d\left[(V_1^l(\lambda),\ldots,V_n^l(\lambda)),(S_1,\ldots,S_n)\right]\to 0} \dfrac{W_F\left[\left(V_1^l(\lambda),\ldots,V_n^l(\lambda)\right),(S_1,\ldots,S_n)\right]}{d\left[(V_1^l(\lambda),\ldots,V_n^l(\lambda)),(S_1,\ldots,S_n)\right]} = 0.$

Proof: The proof to this lemma is given in[5].

Now we are ready to give some properties of a differentiable convex n-set function. The following theorem and lemma, which are extremely important for our study, were proved by Corely[4].

Theorem 2.1

(Theorem 4.5[5]) Let $F : S^n \to R$ be differentiable on a convex subfamily $S^n \subset A^n$. If F is convex, then for all $(R_1,\ldots,R_n),(S_1,\ldots,S_n) \in S^n$

$$F(R_1,\ldots,R_n) - F(S_1,\ldots,S_n) \geq \sum_{i=1}^{n} \left\langle f_*^i, \chi_{R_i} - \chi_{S_i} \right\rangle \tag{2.1}$$

where f_*^i is the i th partial derivative of F at $(S_1,\ldots,S_n)$.

Lemma 2.2

(Theorem 4.6[5]) Let $F : S^n \to R$ be differentiable on a convex subfamily $S^n \subset A^n$. If

$$F(R_1,\ldots,R_n) - F(S_1,\ldots,S_n) \geq \sum_{i=1}^{n} \left\langle f_*^i, \chi_{R_i} - \chi_{S_i} \right\rangle$$

is satisfied for all $(R_1,\ldots,R_n),(S_1,\ldots,S_n) \in S^n$, where f_*^i is the i th partial derivative of F at $(S_1,\ldots,S_n)$, then F is convex.

A geometric interpretation of the above results can be given as follows. For a differentiable convex function F on S^n, the additive n-set function $F(S_1,\ldots,S_n) \geq \sum_{i=1}^{n} \left\langle f_*^i, \chi_{R_i} - \chi_{S_i} \right\rangle$, where $(S_1,...,S_n) \in S^n$, never overestimates $F(R_1,...,R_n)$ for any $(R_1,...,R_n) \in S^n$ and obviously for a concave function F on S^n, the additive n-set function

$$F(S_1,...,S_n) = \sum_{i=1}^{n} \langle f_*^i, \chi_{R_i} - \chi_{S_i} \rangle,$$

where $(S_1,...,S_n) \in S^n$, never underestimates $F(R_1,...,R_n)$ for any $(R_1,...,R_n) \in S^n$.

As an immediate consequence of theorem 2.1 another necessary condition can be stated for a differentiable convex n-set function.

Theorem 2.2

Let $F : S^n \to R$ be differentiable on a convex subfamily $S^n \subset A^n$. A necessary condition that F be convex on S^n is that for each $(R_1,...,R_n),(S_1,...,S_n) \in S^n$

$$\sum_{i=1}^{n} \langle f_S^i - f_R^i, \chi_{S_i} - \chi_{R_i} \rangle \geq 0,$$

where f_S^i and f_R^i are the i th partial derivatives of F at $(S_1,...,S_n)$ and $(R_1,...,R_n)$ respectively.

Proof:

Since F is differentiable at $(S_1,...,S_n)$ and $(R_1,...,R_n)$, by using inequality (2.1) twice we have

$$F(S_1,...,S_n) - F(R_1,...,R_n) \geq \sum_{i=1}^{n} \langle f_R^i, \chi_{S_i} - \chi_{R_i} \rangle \tag{2.2}$$

$$F(R_1,...,R_n) - F(S_1,...,S_n) \geq \sum_{i=1}^{n} \langle f_S^i, \chi_{R_i} - \chi_{S_i} \rangle \tag{2.3}$$

By taking the summation of inequality (2.2) and inequality (2.3) it follows that

$$0 \geq \sum_{i=1}^{n} \langle f_R^i, \chi_{S_i} - \chi_{R_i} \rangle - \sum_{i=1}^{n} \langle f_S^i, \chi_{S_i} - \chi_{R_i} \rangle.$$

That is,

$$\sum_{i=1}^{n} \langle f_S^i - f_R^i, \chi_{S_i} - \chi_{R_i} \rangle \geq 0.$$

Hence, the proof of the theorem is complete.

In order to obtain sufficient conditions for a local minimum, Morris[1] introduced the concept of local convexity of a set function. Following the same line we define the local convexity of an n-set function as follows.

Definition 2.6

A differentiable n-set function $F : A^n \to R$ is locally convex at $(S_1,...,S_n) \in A^n$ if there exist an $\varepsilon > 0$ such that for $(R_1,...,R_n) \in A^n$ satisfying $d[(R_1,..., R_n),(S_1,..., S_n)] < \varepsilon$ implies

$$F(R_1,...,R_n) \geq F(S_1,...,S_n) + \sum_{i=1}^{n} \langle f_*^i, \chi_{R_i} - \chi_{S_i} \rangle.$$

Now we state and prove a sufficient condition for a differentiable locally convex n-set function.

Theorem 2.3

Let $F : A^n \to R$ *be differentiable on* A^n *and* $(S_1,\dots,S_n) \in A^n$. *If there exists a* $\gamma > 0$ *such that*

$$\sum_{i=1}^{n} \left\langle f_S^i - f_R^i, \chi_{S_i} - \chi_{R_i} \right\rangle \geq \gamma d[(R_1,\dots,R_n),(S_1,\dots,S_n)]$$

for any $(R_1,\dots,R_n) \in A^n$ *satisfying* $d[(R_1,\dots,R_n),(S_1,\dots,S_n)] < \varepsilon_1$ *for some* $\varepsilon_1 > 0$, *then F is locally convex at* $(S_1,\dots,S_n)$.

Proof: Since $F : A^n \to R$ is differentiable on A^n, by using definition 2.5 we obtain the following two equations:

$$F(R_1,\dots,R_n) = F(S_1,\dots,S_n) + \sum_{i=1}^{n} \left\langle f_S^i, \chi_{R_i} - \chi_{S_i} \right\rangle + W_F^1[(R_1,\dots,R_n),(S_1,\dots,S_n)] \qquad (2.4)$$

and

$$F(S_1,\dots,S_n) = F(R_1,\dots,R_n) + \sum_{i=1}^{n} \left\langle f_R^i, \chi_{S_i} - \chi_{R_i} \right\rangle + W_F^2[(R_1,\dots,R_n),(S_1,\dots,S_n)] \qquad (2.5)$$

for any $(R_1,\dots,R_n) \in A^n$.

Adding equations (2.4) and (2.5) gives

$$\sum_{i=1}^{n} \left\langle f_S^i - f_R^i, \chi_{S_i} - \chi_{R_i} \right\rangle = W,$$

where $W = W_F^1[(R_1,\dots,R_n),(S_1,\dots,S_n)] + W_F^2[(R_1,\dots,R_n),(S_1,\dots,S_n)]$.

But by the hypothesis

$$\sum_{i=1}^{n} \left\langle f_S^i - f_R^i, \chi_{S_i} - \chi_{R_i} \right\rangle \geq \gamma\, d[(R_1,\dots,R_n),(S_1,\dots,S_n)]$$

whenever $d[(R_1,\dots,R_n),(S_1,\dots,S_n)] < \varepsilon_1$, for some $\varepsilon_1 > 0$. That is,

$$F(R_1,\dots,R_n) \geq F(S_1,\dots,S_n) + \sum_{i=1}^{n} \left\langle f_S^i, \chi_{R_i} - \chi_{S_i} \right\rangle + \gamma\, d[(R_1,\dots,R_n),(S_1,\dots,S_n)]$$
$$- W_F^2[(R_1,\dots,R_n),(S_1,\dots,S_n)] \quad (2.6)$$

since $\lim_{d[(R_1,\dots,R_n)(S_1,\dots,S_n)]\to 0} \dfrac{W_F^2[(R_1,\dots,R_n),(S_1,\dots,S_n)]}{d\left[(R_1,\dots,R_n),(S_1,\dots,S_n)\right]} = 0$, given $\gamma > 0$ there exist an

$\varepsilon_2 > 0$ such that $-\gamma \leq \dfrac{W_F^2[(R_1,\dots,R_n),(S_1,\dots,S_n)]}{d\left[(R_1,\dots,R_n),(S_1,\dots,S_n)\right]} \leq \gamma$

whenever $d[(R_1,\dots,R_n),(S_1,\dots,S_n)] < \varepsilon_2$. That is,

$$\gamma d[(R_1,\dots,R_n),(S_1,\dots,S_n)] - W_F^2[(R_1,\dots,R_n),(S_1,\dots,S_n)] \geq 0$$

whenever $d[(R_1,\dots,R_n),(S_1,\dots,S_n)] < \varepsilon_2$.

Let $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$. then by equation (2.6)

$$F(R_1,\dots,R_n) \geq F(S_1,\dots,S_n) + \sum_{i=1}^{n} \left\langle f_S^i, \chi_{R_i} - \chi_{S_i} \right\rangle$$

whenever $d[(R_1,\dots,R_n),(S_1,\dots,S_n)] < \varepsilon$.

This implies that F is locally convex at $(S_1,\dots,S_n)$.

It is important to mention that all the results can be extended directly to concave functions by changing the inequalities $\geq$ to $\leq$.

Before we conclude this section we give a simple result in mathematical programming with differentiable n–set function.

Theorem 2.4

Let S^n be a convex subfamily of A^n and let $F : S^n \to R$ be a convex $n-$ set function differentiable at $(S_1,\dots,S_n) \in S^n$. Then $(S_1,\dots,S_n)$ solves the minimization problem:

Minimize $F(R_1,\dots,R_n)$

subject to

$$(R_1,\dots,R_n) \in S^n$$

if and only if $\sum_{i=1}^{n} \left\langle f_*^i, \chi_{R_i} - \chi_{S_i} \right\rangle \geq 0$, for all $(R_1,\dots,R_n) \in S^n$, where f_*^i is the i th partial derivative of F at $(S_1,\dots,S_n)$.

Proof: Since F is differentiable at $(S_1,\dots,S_n)$, we have for each $\lambda \in (0,1)$ and $(R_1,\dots,R_n) \in S^n$,

$$F(V_1^l(\lambda),\dots,V_n^l(\lambda) = F(S_1,\dots,S_n) + \sum_{i=1}^{n} \left\langle f_*^i, \chi_{V_i^l(\lambda)} - \chi_{S_i} \right\rangle +$$

$$W_F\left[(V_1^l(\lambda),\dots,V_n^l(\lambda)),(S_1,\dots,S_n\right]$$

for any Morris sequence $\{ V_i^l(\lambda) \} \subset S$ associated with $\langle \lambda, R_i, S_i \rangle$. for each $i = 1,\dots,n$.
Therefore

$$\limsup_{l\to\infty} F(V_1^l(\lambda),\dots,V_n^l(\lambda) \leq F(S_1,\dots,S_n) + \limsup_{l\to\infty} \sum_{i=1}^{n} \left\langle f_*^i, \chi_{V_i^l(\lambda)} - \chi_{S_i} \right\rangle +$$

$$\limsup_{l\to\infty} W_F\left[(V_1^l(\lambda),\dots,V_n^l(\lambda)),(S_1,\dots,S_n\right] .$$

Since $\limsup_{l\to\infty} \sum_{i=1}^{n} \left\langle f_*^i, \chi_{V_i^l(\lambda)} - \chi_{S_i} \right\rangle \leq \lambda \sum_{i=1}^{n} \left\langle f_*^i, \chi_{R_i} - \chi_{S_i} \right\rangle$,
we can establish the following inequality

$$\limsup_{l\to\infty} F(V_1^l(\lambda),\ldots,V_n^l(\lambda) \le F(S_1,\ldots,S_n) + \lambda\sum_{i=1}^{n}\left\langle f_*^i, \chi_{R_i} - \chi_{S_i}\right\rangle +$$

$$\limsup_{l\to\infty} W_F\left[\left(V_1^l(\lambda),\ldots,V_n^l(\lambda)\right),(S_1,\ldots,S_n\right] .$$

$$= F(S_1,\ldots,S_n) + \lambda\sum_{i=1}^{n}\left\langle f_*^i, \chi_{R_i} - \chi_{S_i}\right\rangle + o(\lambda)$$

by lemma 2.1.

Since $F(S_1,\ldots,S_n) \le F(R_1,\ldots,R_n)$ for any $(R_1,\ldots,R_n) \in S^n$, we have

$$0 \le \limsup_{l\to\infty} F(V_1^l(\lambda),\ldots,V_n^l(\lambda) - F(S_1,\ldots,S_n) \le \lambda\sum_{i=1}^{n}\left\langle f_*^i, \chi_{R_i} - \chi_{S_i}\right\rangle + o(\lambda).$$

Therefore

$$\sum_{i=1}^{n}\left\langle f_*^i, \chi_{R_i} - \chi_{S_i}\right\rangle = -\frac{o(\lambda)}{\lambda}$$

Letting $\lambda \to 0$ we obtain the desired result.

To obtain the converse, let us assume

$$\sum_{i=1}^{n}\left\langle f_*^i, \chi_{R_i} - \chi_{S_i}\right\rangle \ge 0 \text{ for any } (R_1,\ldots,R_n) \in S^n.$$

Since F is convex, by theorem 2.1

$$F(R_1,\ldots,R_n) - F(S_1,\ldots,S_n) \ge \sum_{i=1}^{n}\left\langle f_*^i, \chi_{R_i} - \chi_{S_i}\right\rangle.$$

That is, $F(R_1,\ldots,R_n) \ge F(S_1,\ldots,S_n)$. Thus the result.

3. CONCLUSION

Morris[1] defined the concept of convex set functions by means of special class of sequences in analogy with the concept of the convex combinations of two sets. Such sequences are named as Morris sequences by Choi, Hsia and Lee[6]. Until Corely[4] introduced the general theory of n-set functions, most of the work had been confined to set functions. Since Corely[4] introduced the concept of convexity and differentiability of an n-set function in 1987, there has been an extensive interest in the theory of mathematical programming with n-set functions as evidenced by the publication of many papers of the subject. In this paper, our objective was to continue the earlier work which has been done for n-set functions and establish more results in mathematical programming with n-set functions. In this paper, we were able to sate and prove first order necessary and sufficient conditions for a differentiable convex n-set function.

Morris[1] proposed a numerical method to solve convex optimization problems with set functions. But the method was proved to be computationally inefficient. Therefore, there is a greater need for research in the finding of efficient numerical methods to solve convex and nonconvex optimization problems with n-set functions. We hope that the results presented in this paper will stimulate future work in obtaining more results, extending the existing results, and developing numerical methods for convex and nonconvex optimization problems with n-set functions.

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